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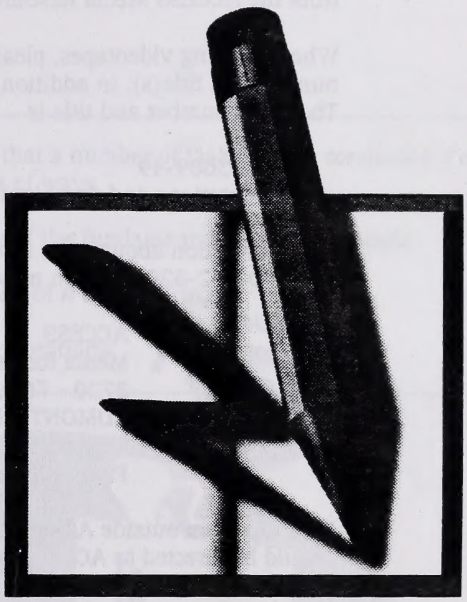
MATH FACTOR

PERMUTATIONS AND COMBINATIONS



Lesson 1: Fundamental Counting Principle and Permutations

Student Workbook

Produced by

THE EDUCATION STATION

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Alberta EDUCATION

This guide accompanies **Fundamental Counting Principle and Permutations** from the Math Factor series.

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Editor: Wilma Gonsalves
Producer: Katherine Rankin

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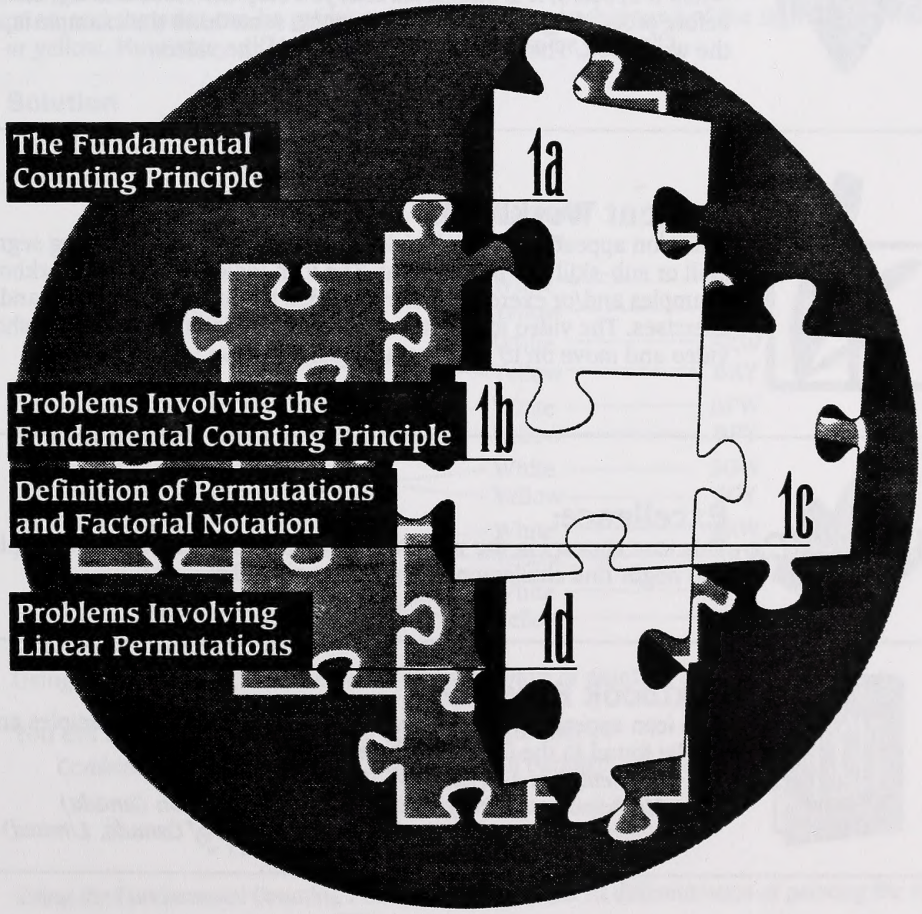
Lesson 1: FUNDAMENTAL COUNTING PRINCIPLE AND PERMUTATIONS

LEARNER EXPECTATIONS

By the end of Lesson 1 you should be able to

- calculate the total number of ways that a number of tasks can be conducted if each task can be performed in a multiple of ways
- solve problems that involve the use of the fundamental counting principle
- calculate the number of permutations of n things using $n!$
- solve problems involving linear permutations

This lesson has four parts:



EXPLANATION OF ICONS

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Video:

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- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

The Fundamental Counting Principle

Challenge yourself

How many different outfits could a person wear if they had two pairs of shoes, four pairs of pants and three shirts?

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Watch the Introduction and Video Segment 1a.

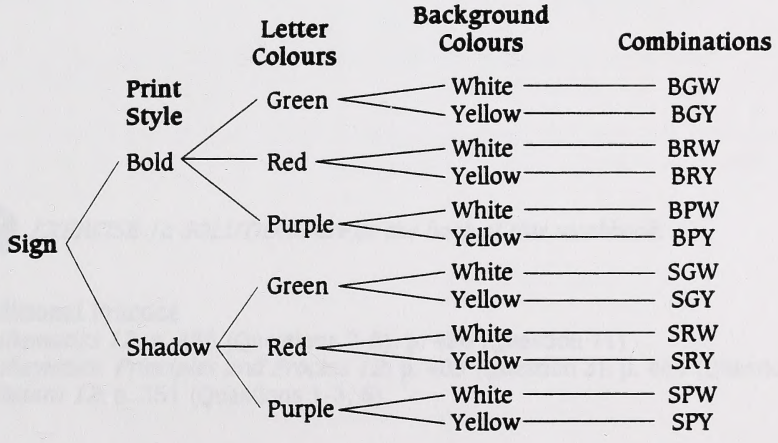
The **Fundamental Counting Principle** states that in a sequence of events, where the first event has "k_1" possibilities and the second event has "k_2" possibilities, and so on, the total number of possible events is $(k_1)\ (k_2)\ (k_3)\ \ldots\ (k_n)$.

Example

The graduation committee wants a sign painted. They can choose either bold or shadow print, and the print can be green, red or purple. The background of the sign can be white or yellow. How many different ways could the sign be painted?

Solution

Use a tree diagram.



Using a tree diagram, the number of different ways of painting the sign is 12.

You can also use the Fundamental Counting Principle.

$$\begin{aligned}\text{Combinations} &= (\text{print style})(\text{letter colours})(\text{background colours}) \\ &= (2)(3)(2) \\ &= 12\end{aligned}$$

Using the Fundamental Counting Principle, there are also 12 different ways of painting the sign.

TEXT

Additional Examples
Mathematics 12: p. 484 (Example 1)
Mathematics: Principles and Process 12: p. 402
Holtmath 12: p. 350

Exercises

1. John has two pairs of shoes, one black and the other brown. He has three dress shirts in white, blue and yellow. He also has dress pants in grey, navy, brown and green. How many different combinations of pants, shirts and pairs of shoes does John have?

2. There are two roads (Highways 14 and 15) connecting Town A and Village B, and four roads (Highways 16, 17, 18 and 19) connecting Village B and City C. Use a diagram to show the number of ways to travel from Town A to City C via Village B.

3. Fast Food offers ten different kinds of hamburgers, six different kinds of drinks and either crispy fries or curly fries. If a meal includes a hamburger, fries and a drink, then how many different meals could be ordered?

4. Sandy has three textbooks: *Math Wonders*, *Chemistry Challenges* and *Physics Impossibilities*. Use a tree diagram to illustrate the number of different ways that the textbooks could be arranged on a bookshelf.

EXERCISE 1a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 485 (Questions 2-5), p. 486 (Question 11)
Mathematics: Principles and Process 12: p. 403 (Question 3), p. 404 (Questions 4, 6)
Holtmath 12: p. 351 (Questions 1-3, 6)

Problems Involving the Fundamental Counting Principle

How many ways can a baseball manager arrange a batting order of nine players?

Watch Video Segment 1b.

Example 1

A mathematics test consists of ten multiple-choice questions. Each question has three answers. How many different sets of answers can be chosen?

Solution

There are three possible answers for each question therefore the total number of different sets of answers that can be chosen is:

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^{10}$$
$$= 59\,049$$

Example 2

Determine the number of different three-letter words that can be formed using the letters **C**, **D**, **E**, **F**, **G** and **A** when:

- no letter can be repeated
- the letters can be repeated

Solution

- if no letter can be repeated

  The first letter can be chosen from any one of the six letters. Having filled this position, the second letter is chosen from the remaining five letters and consequently, the third letter is chosen from the remaining four letters. Thus, the number of three-letter words is $6 \times 5 \times 4 = 120$.

- if the letters can be repeated

  In this case there are six choices for each position. Thus, the number of three-letter words is $6 \times 6 \times 6 = 216$.

Example 3

Determine the number of four-digit numbers, not beginning with zero, that can be formed from the numbers **0**, **1**, **2**, **3**, **4**, **5**, **6**, **7** and **8** when the numbers:

- can be repeated
- cannot be repeated

Solution

- if the numbers can be repeated

  There are four positions, but the first position cannot contain a zero. Thus, there are eight choices for the first position. For each of the remaining three positions there are nine choices.

  $8 \times 9 \times 9 \times 9 = 5832$ numbers

- if the numbers cannot be repeated

  There are four positions, and again the first position cannot contain a zero. Thus, there are eight choices for the first position. This leaves eight remaining numbers (including zero) to choose from for the second position, seven numbers to choose from for the third position and six numbers to choose from for the last position.

  $8 \times 8 \times 7 \times 6 = 2688$ numbers

TEXT

Additional Examples
Mathematics 12: p. 484 (Example 2)
Mathematics: Principles and Process 12: p. 403
Holtmath 12: p. 351

Exercises

5. If numbers can be repeated, then how many three-digit numbers less than 400 can be formed from the numbers 2, 3, 5, 7 and 8? Solve this problem by answering the following:

   a. Can the first digit be 5, 7 or 8? _____

   b. The first digit can be chosen in _____ ways.

   c. The second digit can be chosen in _____ ways.

   d. The third digit can be chosen in _____ ways.

   e. The number of three-digit numbers is ___×___×___ = _____.

6. A hockey stadium has 15 gates: 7 are on the north side and 8 are on the south side.

   a. In how many ways can Peter enter by a south gate and leave by a north gate?

   b. In how many ways can Peter enter and leave the stadium?

7. In how many ways can Mr. and Mrs. MacDonald and their four children line up in order to have their family picture taken?

8. In Red Deer, all telephone numbers contain seven digits and begin with a three. The second digit is never a zero. How many different phone numbers are possible for Red Deer?

9. London, England, recently ran out of available telephone numbers, so it was decided to expand the codes. A two-digit prefix, either 71 or 81, was added to each seven-digit telephone number, doubling the previous quantity of telephone numbers. If the seven-digit number cannot begin with a 0 or a 1, then how many possible telephone numbers now exist for London?

EXERCISE 1b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 485 (Question 8), p. 486 (Questions 9, 10, 12)
Mathematics: Principles and Process 12: p. 404 (Questions 5, 7)
Holtmath 12: p. 351 (Questions 4, 5), p. 352 (Questions 10-12, 15, 16)

Definition of Permutations and Factorial Notation

Challenge yourself

How many different five-digit numbers can be made using the numbers 1 through 5?

MATH FACTOR

Watch Video Segment 1c.

A permutation is defined as an ordered arrangement or sequence of all or part of a set of things; for example, an arrangement of things where order is important.

$$n! = n(n-1)(n-2)(n-3)\ldots(2)(1)$$

$n!$ is read as n factorial.

Example 1

Expand and evaluate 7!.

Check your calculator for the location of the factorial key.

Solution

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Example 2

Expand and evaluate $\frac{8!}{5!}$.

Solution

$$\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$$
$$= 336$$

Example 3

Write the following in factorial notation.

$$11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

Solution

$11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 11!$

Example 4

Write the following in factorial notation.

$9 \times 8 \times 7 \times 6 \times 5!$

Solution

$$9 \times 8 \times 7 \times 6 \times 5! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$
$$= 9!$$

TEXT

Additional Examples
Mathematics 12: p. 487 (Example 2)
Mathematics: Principles and Process 12: p. 403

Exercises

10. Read the following statements and decide if the arrangements are permutations.

    a. There are eight horses in a race. Three horses are selected to place in the top three positions.

    b. There are eight horses in a race. What are the number of different ways that three horses will place first, second or third in the race.

    c. An ice cream parlour has twenty different kinds of ice cream and three different kinds of cones. A person selects a particular cone and a particular kind of ice cream.

    d. Out of a group of thirty people, a president, secretary and treasurer are chosen.

    e. Out of a group of twenty people, three people are chosen to form a committee.

11. Evaluate the following:

    a. $6!$

    b. $10!$

    c. $5! \times 4!$

12. Evaluate the following:

a. $\frac{5!}{2}$

b. $\frac{9!}{4!}$

c. $\frac{5!3!}{2!}$

13. Write the following products in factorial notation.

a. $4 \times 3 \times 2 \times 1$

b. $15 \times 14 \times 13 \times 12!$

c. $n(n-1)(n-2)...(2)(1)$

14. Write $(n+2)(n+1)(n)...(2)(1)$ in factorial notation.

EXERCISE 1c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 489 (Question 1)
Mathematics: Principles and Process 12: p. 403 (Questions 1, 2)
Holtmath 12: p. 354 (Questions 3, 4)

Problems Involving Linear Permutations

How many ways can four different algebra books and three different geometry books be arranged on a shelf if the geometry books must be kept together?

Example 1

Determine the number of five-digit whole numbers that can be formed from the digits 0, 1, 2, 3, 4, 5 and 6 when the:

- five digits are different
- digits can be repeated
- numbers are odd and the digits are not repeated

Solution

- if the five digits are different

  There are five positions and the first position cannot contain a zero; so there are six choices for the first position. This leaves six remaining numbers (including zero) to choose from for the second position, five for the third position and so on.

  $6 \times 6 \times 5 \times 4 \times 3 = 2160$ numbers

- if the digits can be repeated

  There are five positions to fill, and the first cannot be a zero.

  $6 \times 7 \times 7 \times 7 \times 7 = 14\,406$ numbers

- if the numbers are odd and the digits are not repeated

  In order for the number to be odd, only the 1, 3 or 5 can be used in the last position. Once this digit is chosen, five choices remain for the first digit (zero is not a choice), five choices for the second digit, four for the third and three for the fourth digit.

  $5 \times 5 \times 4 \times 3 \times 3 = 900$ numbers

Example 2

A family has two parents and five children, one of which is a baby. If one of the parents has to drive and the baby must be placed in the permanent car seat, then how many ways can the family be arranged in a minivan that has seven seats?

Solution

There are two choices for the driver's seat and only one choice for the car seat (the baby). The remaining seats can be occupied by any one of the five remaining members of the family.

$2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1$ or $2 \times 5! = 240$ ways

TEXT

Additional Examples
Mathematics 12: p. 485 (Example 4)
Mathematics: Principles and Process 12: p. 405

Exercises

15. In Canada, a postal code consists of a letter, a number, a letter, a number, a letter and a number.

    a. If all the numbers 0-9 and 26 letters could be used, then what is the total possible number of postal codes that exist in Canada?

    b. All postal codes in Alberta start with the letter T. If all the numbers 0-9 and 26 letters could be used for the choices in the remaining five positions, then how many codes are possible in Alberta?

    c. At present, only 20 letters are used; D, F, I, O, Q and U are omitted. Using only 20 letters and the numbers 0-9, what is the total possible number of postal codes that could exist in Canada? In Alberta?

16. If repetition of letters and numbers is not allowed, then how many different licence plates can be made using three letters followed by three numbers from the numbers 1, 2, 3, ..., 8, 9?

17. Determine the number of three-digit odd numbers between 200 and 600 that can be formed from the digits 0, 1, 2, 4, 7 and 9 when:

    a. the digits cannot be repeated

    b. repetition of digits is allowed

18. How many ways can five different-coloured cups and three different-coloured plates be arranged in a line on a shelf if the plates must be kept together?

EXERCISE 1d SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 485 (Questions 6, 7), p. 486 (Questions 13, 14, 18)
Mathematics: Principles and Process 12: p. 405 (Questions 8, 10-12)
Holtmath 12: p. 352 (Questions 9, 13, 14, 19)

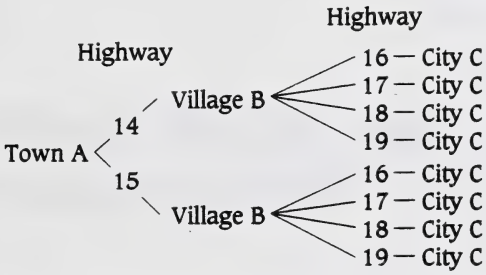
WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. The Fundamental Counting Principle and the concept of permutations are the bases of **Lesson 2: Permutations with Restrictions**.

Solutions

EXERCISE 1a

1. Different ways $= (\text{shoes})(\text{shirts})(\text{pants})$
$= (2)(3)(4)$
$= 24$

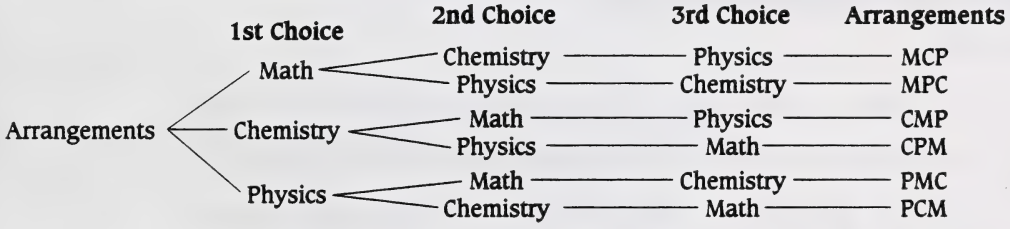
2.



There are eight different routes that can be taken.

3. Meals $= (\text{hamburgers})(\text{drinks})(\text{fries})$
$= (10)(6)(2)$
$= 120$

4.



There are six different ways to arrange the textbooks.

EXERCISE 1b

5. a. no

b. 2

c. 5

d. 5

e. $2 \times 5 \times 5 = 50$

6. a. $8 \times 7 = 56$ ways

b. $15 \times 15 = 225$ ways

7. $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways

8. $1 \times 9 \times 10 \times 10 \times 10 \times 10 \times 10 = 900\,000$ different phone numbers

9. $2 \times 1 \times 8 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 16\,000\,000$ different phone numbers

EXERCISE 1c

10. a. no b. permutation c. no

d. permutation e. no

11. a. $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$= 720$

b. $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$= 3\,628\,800$

c. $5! \times 4! = 5 \times 4 \times 3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1$
$= 2880$

12. a. $\frac{5!}{2} = \frac{(5 \times 4 \times 3 \times 2 \times 1)}{2}$
$= 60$

b. $\frac{9!}{4!} = \frac{(9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)}{(4 \times 3 \times 2 \times 1)}$
$= 15\,120$

c. $\frac{5!3!}{2!} = \frac{(5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1)}{(2 \times 1)}$
$= 360$

13. a. $4!$ b. $15!$ c. $n!$

14. $(n + 2)!$

EXERCISE 1d

15. a. $26 \times 10 \times 26 \times 10 \times 26 \times 10 = 17\,576\,000$ postal codes could exist in Canada.

b. $1 \times 10 \times 26 \times 10 \times 26 \times 10 = 676\,000$ postal codes could exist in Alberta.

c. $20 \times 10 \times 20 \times 10 \times 20 \times 10 = 8\,000\,000$ postal codes could exist in Canada.

$1 \times 10 \times 20 \times 10 \times 20 \times 10 = 400\,000$ postal codes could exist in Alberta.

16. $26 \times 25 \times 24 \times 9 \times 8 \times 7 = 7\,862\,400$ different licence plates

17. a. In order for the number to be between 200 and 600, the first digit can be 2 or 4. For the number to be odd, the last digit can be 1, 7 or 9. Once these digits are chosen, there are four choices left. Therefore, the number of different numbers = $2 \times 4 \times 3 = 24$.

b. There are two choices for the first digit, six choices for the second digit and three choices for the third digit. Therefore, the number of different numbers = $2 \times 6 \times 3 = 36$.

18. The plates must be kept together, so consider them as one. Then there are six different objects to be arranged, or 6! arrangements, but the three different plates can be arranged in 3! different ways. Therefore, the number of different arrangments = $6! \times 3! = 4320$ ways.

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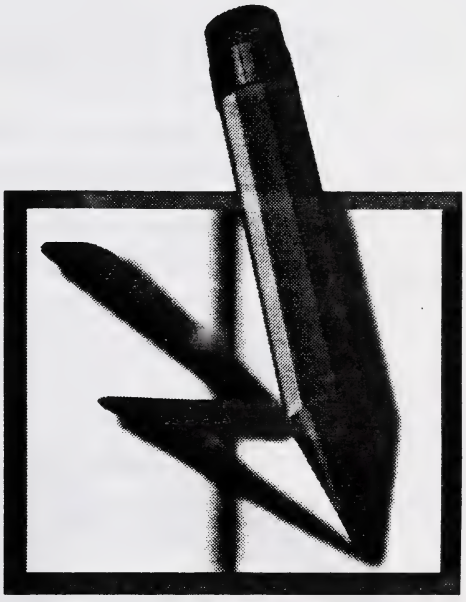
MATH FACTOR

PERMUTATIONS AND COMBINATIONS



Lesson 2:
Permutations with Restrictions

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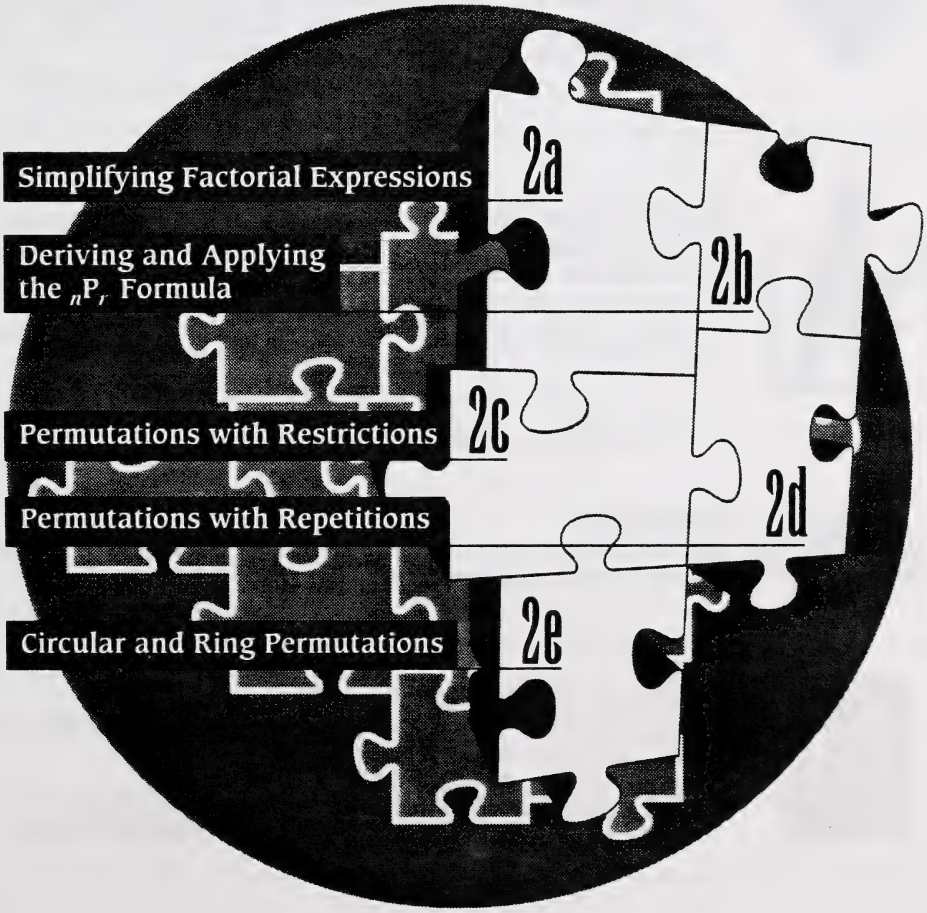
Lesson 2: PERMUTATIONS WITH RESTRICTIONS

LEARNER EXPECTATIONS

By the end of Lesson 2 you should be able to

- demonstrate an understanding that a permutation is an ordered arrangement in which certain restrictions can be placed
- simplify arithmetic and algebraic factorial expressions
- calculate the number of permutations of n things taken r at a time, using a permutation formula or a calculator
- solve problems involving permutations with repetitions and circular and ring permutations
- demonstrate the ability to distinguish between the different types of permutation problems

This lesson has five parts:



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Simplifying Factorial Expressions

Watch the Introduction and Video Segment 2a.

When working with permutations it is necessary to be able to simplify expressions which contain arithmetic and algebraic factorials.

In the video segment, the statement is made that "n factorial represents the **product** of all the consecutive numbers from n down to 1." This can be written in equation form as $n! = n(n-1)(n-2)...(3)(2)(1)$.

Arithmetic expressions that involve factorials can be easily evaluated by calculator or by applying steps similar to those in the following examples.

Example 1

Evaluate $\dfrac{10!}{(10-2)!}$.

Solution

Method 1

$$\frac{10!}{(10-2)!} = \frac{10!}{8!}$$

$$= \frac{(10 \times 9 \times 8!)}{8!}$$

$$= 10 \times 9$$

$$= 90$$

by reducing the 8! factors

Method 2

$$\frac{10!}{(10-2)!} = \frac{10(10-1)(10-2)!}{(10-2)!}$$

$$= 10(10-1)$$

$$= 10(9)$$

$$= 90$$

Algebraic expressions that contain factorials can often be simplified to a form that does not contain the factorial symbol.

Example 2

Write $\frac{(n+4)!}{(n+2)!}$ as a trinomial without factorials.

Solution

$$\frac{(n+4)!}{(n+2)!} = \frac{(n+4)(n+3)(n+2)!}{(n+2)!}$$

$$= (n+4)(n+3)$$ by reducing the $(n+2)!$ factors in expanded form

$$= n^2 + 7n + 12$$

Exercises

1. Find the value of $\frac{12!}{9!2!}$.

2. Find the value of $\frac{4!}{5!2!} + \frac{6!}{3!2!}$.

3. Write $\frac{(n-r)!}{(n-r-2)!}$ in a factored form without factorials.

4. Write $\dfrac{(n-1)!\,2!}{(n-3)!}$ in a form without either factors or factorials.

5. Write $(n-4)(n-5)(n-6)!$ as a single factorial.

EXERCISE 2a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 489 (Questions 1, 8, 9)
Mathematics: Principles and Process 12: p. 412 (Question 1)
Holtmath 12: p. 354 (Questions 4, 5)

Deriving and Applying the ${}_nP_r$ Formula

Challenge yourself

From five different horses, how many ways can three of them be lined up for a trail ride?

MATH FACTOR

It is important to remember that in an arrangement or a permutation of things, a **change in order** represents a **different** permutation.

Permutations

A permutation of n objects, taken r objects at a time, is written ${}_nP_r$ and is given as follows:

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Note: The formula by itself does not apply when there are restrictions or repetitions.

Example 1

How many ways can five books be arranged on a shelf if they are to be chosen from nine books?

Solution

Method 1

Use ${}_nP_r$, where $n = 9$ and $r = 5$.

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

$$ {}_9P_5 = \frac{9!}{(9-5)!} $$

$$ = \frac{9!}{4!} $$

$$ = 15\,120 $$

Method 2

Use the Fundamental Counting Principle.

$$ 9 \times 8 \times 7 \times 6 \times 5 = 15\,120 $$

Example 2

Five horses and three ponies are available for a trail ride. In how many ways can three of the horses and then two of the ponies following them be arranged in single file for a trail ride?

Solution

Method 1

Use the Fundamental Counting Principle.

number of horse arrangements × number of pony arrangements
$= (5 \times 4 \times 3) \times (4 \times 3) = 720$

Method 2

Use ${}_nP_r$.

number of horse permutations × number of pony permutations
$= {}_5P_3 \times {}_4P_2$
$= \frac{5!}{(5-3)!} \times \frac{4!}{(4-2)!}$
$= \frac{5!4!}{2!2!}$
$= 720$

TEXT

Additional Examples
Mathematics 12: p. 487 (Examples 1-4)
Mathematics: Principles and Process 12: p. 408
Holtmath 12: p. 353 (Examples 1, 3)

Exercises

6. Write ${}_3P_2 \times {}_5P_4$ as a product of factorials.

7. Find the value of ${}_5P_3 \times 3! - {}_5P_3 \times 2!$.

8. How many different six-digit ID codes are possible if no digit is to be used more than once?

9. How many four-letter words can be formed from the letters of the word **HEXAGON** if no letter can be used more than once?

EXERCISE 2b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 489 (Questions 10-13)
Mathematics: Principles and Process 12: p. 409 (Questions 1-3)
Holtmath 12: p. 354 (Questions 2, 6)

Permutations with Restrictions

Challenge yourself

Four horses are lined up for a trail ride. In how many ways can this be done if two particular horses must be adjacent?

MATH FACTOR

Watch Video Segment 2c.

Restrictions are sometimes placed on how objects are to be arranged. In these cases, a good approach is to **deal with the restrictions first.**

Example

For each of the following, determine the number of permutations of the letters in the word **OBJECTS**:

- if no restrictions apply
- if the last two letters must be consonants
- if the last letter must be an S
- if B, J and T must be together in any order

Solution

- if no restrictions apply
  All seven letters are different. So, there are $7! = 5040$ arrangements.

- if the last two letters must be consonants
  First, for the last two letters, pick two of the five consonants.

  $${}_5P_2 = \frac{5!}{3!} = 20$$

  The five remaining letters can be arranged in 5! ways. Altogether, there are $20 \times 5! = 2400$ arrangements.

- if the last letter must be an S
  The last letter must be S, so there are six other letters to arrange. This can be done in $6! = 720$ ways.

- if B, J and T must be together in any order
  Consider B, J and T as a single symbol. This BJT symbol and the four other letters can be arranged in $(1 + 4)! = 5!$ ways. B, J and T themselves can be arranged in 3! ways. Altogether, there are $3! \times 5! = 720$ permutations.

TEXT

Additional Examples
Mathematics 12: p. 491 (Examples 1, 2)
Holtmath 12: p. 354 (Examples 4, 6)

Exercises

10. For each of the following, determine the number of permutations of the letters in the word **COMPUTER**.

    a. if the middle two letters must be consonants

    b. if a vowel must be at each end

11. From a group of three males and six females, five are selected and then seated in a row of five chairs. If two females must occupy the first two chairs and a male must sit next to the second female, in how many ways can this be done?

12. For each of the following, in how many ways can the letters A, E, I, O, U, P, Q, R and S be arranged?

    a. if no two consonants are together

    b. if no two vowels are together

EXERCISE 2c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 491 (Questions 1, 4, 5, 7, 9)
Mathematics: Principles and Process 12: p. 409 (Questions 5, 6)
Holtmath 12: p. 355 (Questions 7-9)

Permutations with Repetitions

Watch Video Segment 2d.

Repeated objects are considered to be indistinguishable from one another, so they can be interchanged without forming a new permutation. When you calculate the number of permutations in these cases, you can generalize the patterns that occur into formula form.

Permutation with Repetitions

The number of arrangements of n objects, where a of one kind are alike, b of another kind are alike, c of another kind are alike and so on, is as follows:

$$\frac{n!}{a!b!c!...}$$

The preceding formula can be used to solve a favourite permutation question that deals with repetitions.

Example 1

How many permutations are there for all the letters in the word **MISSISSIPPI**?

Solution

First, regroup the letters as M IIII SSSS PP. So, $n = 11$, $a = 1$, $b = 4$, $c = 4$ and $d = 2$.

$$\frac{n!}{a!b!c!d!} = \frac{11!}{1!4!4!2!}$$
$$= 34\,650$$

Example 2

There is more than one method for solving a related problem. How many permutations of the letters of **MISSISSIPPI** are there that begin with P?

Solution

Method 1

One P must be in first position. The remaining letters to arrange are M IIII SSSS P.
So, $n = 10$, $a = 1$, $b = 4$, $c = 4$ and $d = 1$.

$$\frac{10!}{1!4!4!1!} = 6300$$

Method 2

The number of permutations of all 11 letters = 34 650. Two of the 11 letters are P's. So, logically, $\frac{2}{11}$ of the permutations begin with P.

$$\frac{2}{11} \times 34\,650 = 6300$$

TEXT

Additional Examples
Mathematics 12: p. 493, p. 494 (Example 1)
Holtmath 12: p. 356 (Examples 1-3)

Exercises

13. How many different nine-letter code words can be made from the letters A, A, A, B, B, B, C, C, C?

14. Find the number of distinguishable permutations that can be formed from the letters of the word **CONDOMINIUM**.

15. In question 14, how many of the permutations begin with the letter O?

EXERCISE 2d SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 495 (Questions 2, 4, 5)
Holtmath 12: p. 356 (Questions 1-4, 6)

Circular and Ring Permutations

Challenge yourself

A midway crew is securing six horses on a merry-go-round. How many different arrangements are possible?

Watch Video Segment 2e.

For circular permutations, a handy approach is to first **fix the position of any one of the objects**, and then find the number of (linear) permutations for the remaining objects.

Circular Permutations

The number of circular arrangements of n distinct objects, in which only relative position is involved, is $(n-1)!$.

For ring (reflection) permutations, the number of permutations must be halved. So, for n objects there are $\frac{(n-1)!}{2}$ ring permutations.

Example 1

In how many ways can ten different keys be strung together on a key ring?

Solution

$n = 10$

$$\frac{(n-1)!}{2} = \frac{(10-1)!}{2}$$
$$= \frac{9!}{2}$$
$$= \frac{362\,880}{2}$$
$$= 181\,440$$

Example 2

Jack invites Jill and six other guests to a party. In how many ways can they all be seated in a circle if only relative position is considered and Jack and Jill always sit next to each other?

Solution

First, Jack and Jill are considered to be a single object. So, there are 6 + 1 = 7 objects to arrange in a circle. This can be done in $(7-1)! = 6!$ ways.

But Jack and Jill themselves can be arranged two ways—Jack to the left of Jill or vice versa. So, the total number of arrangements is $2 \times 6! = 2 \times 720 = 1440$.

Additional Examples
Mathematics 12: p. 494 (Question 2)
Mathematics: Principles and Process 12: p. 410
Holtmath 12: p. 357

Exercises

16. To play a game, Jack, Jill and six other children hold hands to form a circle. In how many ways can this be done if Jack and Jill must not stand together?

17. At the front of a banquet room, a circular table is set up for ten special guests. In how many ways can these guests be seated, relative to the other people seated in the banquet room?

EXERCISE 2e SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 496 (Questions 6, 7, 9, 10, 12)
Mathematics: Principles and Process 12: p. 410 (Questions 8-10)
Holtmath 12: p. 357 (Questions 1-5)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of the lesson and check off the points that you have mastered. Go back to the video and study those parts that you find difficult. You should be familiar with methods in this lesson before continuing on to **Lesson 3: Combinations**.

Solutions

EXERCISE 2a

1. $\dfrac{12!}{(9!2!)} = \dfrac{(12\times 11\times 10\times 9!)}{9!\times 2!}$

$= 6\times 11\times 10$

$= 660$

2. $\dfrac{4!}{(5!2!)} + \dfrac{6!}{(3!2!)} = \dfrac{4!}{(5\times 4!\times 2)} + \dfrac{(6\times 5\times 4\times 3!)}{(3!\times 2)}$

$= 0.1 + \dfrac{(6\times 5\times 4)}{2}$

$= 0.1 + 60$

$= 60.1$

3. $\dfrac{(n-r)!}{(n-r-2)!} = \dfrac{(n-r)(n-r-1)(n-r-2)!}{(n-r-2)!}$

$= (n-r)(n-r-1)$

4. $\dfrac{(n-1)!2!}{(n-3)!} = \dfrac{(n-1)(n-2)(n-3)!(2)}{(n-3)!}$

$= (n-1)(n-2)(2)$

$= (n^2 - 3n + 2)(2)$

$= 2n^2 - 6n + 4$

5. $(n-4)(n-5)(n-6)! = (n-4)(n-5)(n-6)\ldots 1$

$= (n-4)!$

EXERCISE 2b

6. $\dfrac{3!}{1!}\times\dfrac{5!}{1!} = 3!\times 5!$

7. ${}_5P_3\times 3! - {}_5P_3\times 2! = {}_5P_3(3! - 2!)$

$= {}_5P_3(6-2)$

$= {}_5P_3(4)$

$= 240$

8. There are ten different digits. Six are to be used to make the code.

Use FCP.

$10\times 9\times 8\times 7\times 6\times 5 = 151\ 200$

Use ${}_nP_r$.

${}_{10}P_6 = \dfrac{10!}{(10-6)!}$

9. There are seven different letters. Four are to be used to make each word.

Use ${}_nP_r$.

${}_7P_4 = \dfrac{7!}{(7-4)!}$

$= \dfrac{7!}{3!}$

$= 840$

Use FCP.

$7\times 6\times 5\times 4 = 840$

EXERCISE 2c

10. a. There are eight letters in all: five consonants and three vowels.

The middle two consonants can be arranged in ${}_5P_2$ ways.

There are six letters remaining to be arranged. This can be done in 6! ways. So, the total number of permutations is as follows:

$= {}_5P_2\times 6!$

$= 20\times 720$

$= 14\ 400$

b. The first and last letter positions can be filled in $3\times 2 = 6$ ways. The other positions can be filled in $(8-2)! = 6!$ ways.

So, the total number of permutations is as follows:

$= 6\times 6!$

$= 6\times 720$

$= 4320$

11. There are ${}_6P_2$ ways of seating two females in the first two seats. Any one of the three males can sit next to them.

From the original pool of nine persons, six remain to be seated. This can be done in ${}_6P_2$ ways.

Altogether, the number of possible arrangements $= {}_6P_2 \times 3 \times {}_6P_2$

$$= ({}_6P_2)^2 \times 3$$

$$= \left(\frac{6!}{4!}\right)^2 \times 3$$

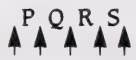
$$= 2700$$

12. a. The vowels can be arranged in 5! ways.

There are six possible positions for the four consonants. They can be arranged ${}_6P_4$ ways.

Therefore, the total number of arrangements is $5! \times {}_6P_4 = 43\,200$.

b. The consonants can be arranged in 4! ways.



There are five possible positions for the five vowels. They can be arranged in 5! ways.

Therefore, the total number of arrangements is $5! \times 4! = 2880$.

EXERCISE 2d

13. $n = 9, a = 3, b = 3, c = 3$

$$\frac{n!}{a!b!c!} = \frac{9!}{3!3!3!}$$

$$= \frac{9!}{(3!)^3}$$

$$= 1680$$

14. Regrouped, the letters are as follows:

C OO NN D II MM U

$$\frac{11!}{2!2!2!2!} = \frac{11!}{(2!)^4}$$

$$= 2\,494\,800$$

15. $\frac{10!}{(2!)^4} = 453\,600$

EXERCISE 2e

16. There are eight children. They can be arranged in a circle in $(8-1)! = 7!$ ways.

There are 1440 arrangements with Jack and Jill adjacent to one another. (Refer to Example 2.)

So, the number of arrangements where they are not adjacent $= 7! - 1440$

$$= 3600$$

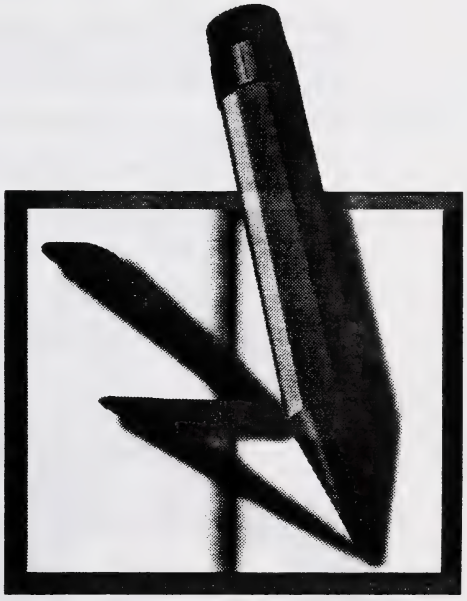
17. In this case, each circular permutation, if rotated, results in a new permutation. So, this is in effect a linear permutation situation.

Ten guests can be arranged in $10! = 3\,628\,800$ ways.

L111 .S2c2 v3

MATH FACTOR

PERMUTATIONS AND COMBINATIONS



Lesson 3: Combinations

Student Workbook

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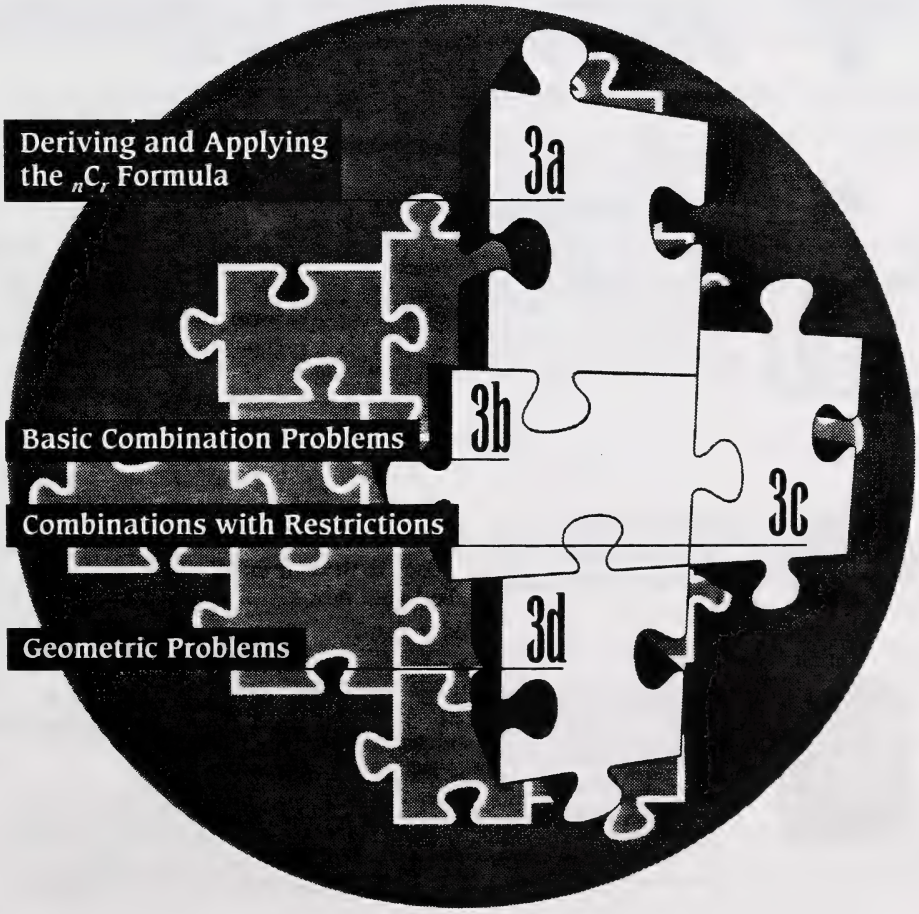
Lesson 3: COMBINATIONS

LEARNER PECTATIONS

By the end of on 3 you should be able to

- demonstrate an understanding that a combination is an arrangement in which order is not important
- calculate the number of combinations of n things taken r at a time, using a formula, calculator or computer
- compare and contrast permutations and combinations
- solve combination problems involving restrictions and geometric shapes

is lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Deriving and Applying the ${}_nC_r$ Formula

Challenge yourself

In how many different ways can three colours be selected from five different colours?

MATH FACTOR

Watch the Introduction and Video Segment 3a.

Example

Video Segment 3a presents a trial-and-error method for finding the number of three-colour combinations for five different-coloured pairs of pants. There is another method that can be used.

Solution

Choosing any three colours is the same as rejecting the two remaining colours. So, to get the answer, you need only count the number of ways of rejecting two colours.

Let the five colours involved be R (red), Y (yellow), B (blue), G (green) and N (navy).
{R, Y, B, G, N} is the set of the five colours.
From {R, Y, B, G, N}: RY, RB, RG, RN are rejected pairs that contain R.
From {Y, B, G, N}: YB, YG, YN are rejected pairs that contain Y.
From {B, G, N}: BG, BN are rejected pairs that contain B.
From {G, N}: GN is a rejected pair that contains G.

Therefore, the number of rejected pairs is ten. So, the number of three-colour combinations is ten.

The pairs can visually be identified and counted within the set {R, Y, B, G, N} without the need to write them down.

Combinations

The number of combinations (where order is not important) of n objects taken r at a time is given by the following:

$$ {}_nC_r = \frac{n!}{(n-r)!r!} $$

Note: The formula does not apply unless the objects are distinct and there are no repetitions.

TEXT

Additional Examples
Mathematics 12: p. 497 (Example 1)
Mathematics: Principles and Process 12: p. 412
Holtmath 12: p. 359 (Example 1)

Exercises

1. In how many ways can five items be chosen from a total of eight different items?

2. In how many ways can three items be excluded when five items are chosen from eight different items?

3. A researcher decides to randomly select 5 of the largest 15 cities in a country for a study. Write an expression that will describe the number of different combinations of 5 cities that are possible.

EXERCISE 3a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 499 (Questions 1-5)
Mathematics: Principles and Process 12: p. 412 (Questions 2, 5)
Holtmath 12: p. 360 (Question 7)

Basic Combination Problems

Challenge yourself

A school debating team consists of five women and four men. How many ways can a group of three be chosen to represent the team in a competition?

MATH FACTOR

Watch Video Segment 3b.

The number of combinations of 16 objects chosen 4 at a time is written ${}_{16}C_4$. An alternate notation for writing this is $\binom{16}{4}$. Both of these notations are read as "sixteen choose four."

Example 1

Find the value of $\binom{10}{4}+\binom{10}{6}$.

Solution

$$\binom{10}{4}=\frac{10!}{(10-4)!4!} \qquad \binom{10}{6}=\frac{10!}{(10-6)!6!}$$

$$=\frac{10!}{6!4!} \qquad =\frac{10!}{4!6!}$$

$$=210 \qquad =210$$

$$\therefore \binom{10}{4}+\binom{10}{6}=210+210$$

$$=420$$

Example 2

John decides to give half of the eight cassettes he doesn't want to each of two friends. In how many ways can this be done? Write the answer using $\binom{n}{r}$ notation.

Solution

A set of four cassettes can be chosen in $\binom{8}{4}$ ways.

Each time a set of four cassettes is selected for one friend, a corresponding set of four is automatically selected for the other friend.

So, the number of ways of giving away the cassettes is $\binom{8}{4}$.

TEXT

Additional Examples
Mathematics 12: p. 498 (Example 2)
Mathematics: Principles and Process 12: p. 412
Holtmath 12: p. 359 (Examples 2, 3)

Exercises

4. a. In how many ways can 3 cards be selected from a standard deck of 52 cards?

   b. How many of the combinations in question 4a consist of 3 cards of the same suit?

5. In how many ways can ten different books be divided equally between A and B? State your answer using $\binom{n}{r}$ notation.

6. Ten men and ten women are being split into groups of four (foursomes) for a golf tournament. Which of the following expressions describe the number of possible foursomes that consist of two men and two women.

   A. ${}_{10}C_2$

   B. ${}_{20}C_2$

   C. ${}_{10}C_2 + {}_{10}C_2$

   D. $({}_{10}C_2)^2$

7. In how many ways can ten books be divided into groups of five, three and two?

EXERCISE 3b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 499 (Questions 4, 5)
Mathematics: Principles and Process 12: p. 413 (Questions 3, 5)
Holtmath 12: p. 360 (Questions 14, 18)

Combinations with Restrictions

Challenge yourself

Ten people are on a students' council. There are four females and six males. A committee of four council members with at most one female is to be chosen. How many different committees could be formed?

MATH FACTOR

Watch Video Segment 3c.

Restrictions are sometimes placed on how objects are to be selected. In these cases, a good approach is to **deal with the restrictions first.**

Example

Three balls are chosen from an urn containing six white, six red and six black balls. Each set of six balls is numbered 1 through 6. For the following, determine the number of different selections of three balls that are possible:

- if there are no restrictions
- if all three balls have the same number
- if exactly two of the balls have the same number

Solution

- if there are no restrictions

  There are 18 balls in total from which 3 are chosen.

$$
\begin{aligned}
{}_{18}C_3 &= \frac{18!}{15!3!} \\
&= 816
\end{aligned}
$$

- if all three balls have the same number

  There is only one way to choose the three number 1 balls. The same is true for the other five numbers. So, there are $6 \times 1 = 6$ selections in total.

- if exactly two of the balls have the same number

  Each same two-number selection can be made ${}_3C_2$ ways. A third number must then be chosen from the 15 balls bearing other numbers. So, altogether there are $6 \times {}_3C_2 \times {}_{15}C_1 = 270$ different selections.

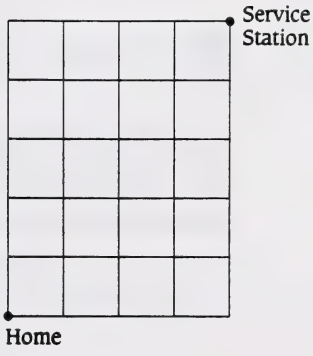
TEXT

Additional Examples
Mathematics 12: p. 498 (Example 3)
Holtmath 12: p. 359 (Example 4)

Exercises

8. In the example for 3c, how many different selections are there if all the balls must be of a different colour?

9. Wilma has a part-time job at a service station five blocks north and four blocks east of where she lives. The blocks in her neighbourhood are all laid out in a rectangular north-south and east-west pattern. How many different routes can she take to walk to work if she walks only nine blocks?



10. For each of the following, determine the number of ways a coach can choose a team of five from eight players:

    a. if Gordon must be included

    b. if Gordon must not be included

11. A committee of four is selected from eight people. What fraction of the possible committees includes two specified people?

EXERCISE 3c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 499 (Questions 6, 9, 12)
Mathematics: Principles and Process 12: p. 413 (Question 7)
Holtmath 12: p. 360 (Questions 12-14)

Geometric Problems

Challenge yourself

Given five points, no three of which lie on the same line, how many line segments, triangles and quadrilaterals can be formed?

MATH FACTOR

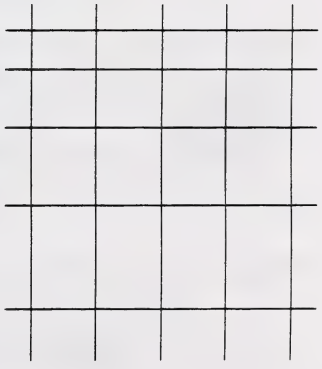
Watch Video Segment 3d.

If you are given n points, no three of which are collinear then an expression for:

- the number of possible line segments joining two points is ${}_nC_2$.
- the number of possible triangles that can be formed is ${}_nC_3$.
- the number of possible quadrilaterals that can be formed is ${}_nC_4$.
- the number of possible polygons of p sides that can be formed is ${}_nC_p$.
- the number of diagonals that can be drawn in a polygon of n sides is ${}_nC_2 - n$.

Example

How many different rectangles are determined by the five horizontal and five vertical line segments?



Solution

Two of the horizontal lines can be chosen in ${}_5C_2$ ways. Similarly, two vertical lines can be chosen ${}_5C_2$ ways.

So, the number of possible rectangles is as follows:

$$\begin{aligned} {}_5C_2 \times {}_5C_2 &= \left({}_5C_2\right)^2 \\ &= \left[\frac{5!}{(3!2!)}\right]^2 \\ &= 100 \end{aligned}$$

Exercises

12. There are ten distinct points on the circumference of the following circle. Using these points, answer the following:

a. How many different line segments can be drawn?

b. How many different octagons can be drawn?

c. How many different p-sided polygons can be drawn?

13. What is the relationship between the answers to questions 12a and 12b? Explain.

EXCELLENCE

14. A polygon of n sides has 44 diagonals. Find n.

EXERCISE 3d SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 500 (Questions 13-15)
Holtmath 12: p. 360 (Questions 9, 15, 21)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of the lesson and check off the points that you have mastered. Go back to the video and study those parts that you find difficult. You should be familiar with the methods used in this lesson before continuing on to **Lesson 4: Probability**.

Solutions

EXERCISE 3a

1. Since this is a combination problem, the answer is as follows:

$$
\begin{aligned}
{}_8C_5 &= \frac{8!}{(8-5)!5!} \\
&= \frac{8!}{3!5!} \\
&= 56
\end{aligned}
$$

2. Every combination of three objects not selected corresponds to a combination of five objects selected.

$$
\begin{aligned}
\therefore {}_8C_3 &= {}_8C_5 \\
&= 56
\end{aligned}
$$

3.
$$
\begin{aligned}
{}_nC_r &= {}_{15}C_5 \\
&= \frac{15!}{(15-5)!5!} \\
&= \frac{15!}{10!5!}
\end{aligned}
$$

EXERCISE 3b

4. a. There are 52 different cards. Three can be chosen in ${}_{52}C_3$ ways.

$$
\begin{aligned}
{}_{52}C_3 &= \frac{52!}{49!3!} \\
&= 22\,100
\end{aligned}
$$

   b. Each suit contains 13 cards. The number of ways of choosing 3 from any suit is ${}_{13}C_3$. Because there are 4 suits, the total number of combinations is $4 \times {}_{13}C_3 = 286$.

5. A's books can be chosen in $\binom{10}{5}$ ways.

   Each choice for A results in a corresponding choice for B. So, the number of ways is $\binom{10}{5}$.

6. D

   Two women can be chosen in ${}_{10}C_2$ ways.
   Two men can also be chosen in ${}_{10}C_2$ ways.
   So, the foursome can be chosen in
   ${}_{10}C_2 \times {}_{10}C_2 = ({}_{10}C_2)^2$ ways.

7. There are ${}_{10}C_5$ ways of choosing five books. Once they are chosen, there are ${}_5C_3$ ways of choosing three books.

   There are ${}_2C_2$ ways of choosing the remaining two books.

   Therefore, there are a total of
   ${}_{10}C_5 \times {}_5C_3 \times {}_2C_2 = 2520$ ways.

EXERCISE 3c

8. The white ball can be chosen in ${}_6C_1 = 6$ different ways. The same is true for the other three colours.

   So, altogether there are $6 \times 6 \times 6 = 216$ different selections.

9. There are several available methods that can be used to arrive at a solution. One of these involves using the diagram itself to work out all the possible routes. A second method involves calculating permutations in which repetitions are involved. In a third method, which is used here, the ${}_nC_r$ formula is applied.

   Let N represent walking one block north and E represent walking one block east. Wilma must walk nine blocks: five blocks to the north and four blocks to the east. One such path, NENENENEN, represents alternately walking north and then east for the total of nine blocks walked.

   To find the total number of routes, find the number of ways of choosing the five N's from the nine blocks that must be walked. So, the total number of routes is ${}_9C_5 = 126$.

10. a. Gordon must be included; therefore, four are chosen from seven = ${}_7C_4$.

    b. Gordon must be excluded; therefore, five are chosen from seven = ${}_7C_5$.

11. There are ${}_8C_4$ or 70 ways of choosing four from eight. If two specified people are included, then there are ${}_6C_2$ or 15 ways.

$$\therefore \frac{15}{70} = \frac{3}{14}$$

EXERCISE 3d

12. a. There are ten points, two of which are chosen for each line segment.

$$\begin{aligned} {}_{10}C_2 &= \frac{10!}{8!2!} \\ &= 45 \end{aligned}$$

    b. There are ten points, eight of which are chosen for each octagon.

$$\begin{aligned} {}_{10}C_8 &= \frac{10!}{2!8!} \\ &= 45 \end{aligned}$$

    c. There are ten points, p of which are chosen for each polygon. So, there are ${}_{10}C_p$ possibilities.

13. The number of line segments (two points chosen) and the number of octagons (eight points chosen) is the same: ${}_{10}C_2 = {}_{10}C_8$. Every time two points are chosen, a corresponding eight points are rejected. So, there is a one-to-one correspondence between these two geometric forms.

14. The number of diagonals is ${}_nC_2 - n$.

$$\begin{aligned} \therefore {}_nC_2 - n &= 44 \\ \frac{n!}{(n-2)!2!} - n &= 44 \\ \frac{(n)(n-1)(n-2)!}{(n-2)!2!} - n &= 44 \\ \frac{n(n-1)}{2} - \frac{n}{1} &= 44 \\ n^2 - n - 2n &= 88 \\ n^2 - 3n - 88 &= 0 \\ (n-11)(n+8) &= 0 \end{aligned}$$

$n - 11 = 0$ or $n + 8 = 0$

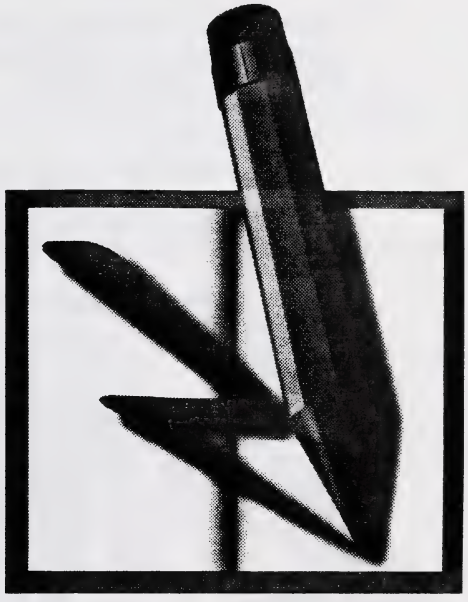
$n = 11$ $\quad$ $n = -8$ (extraneous)

Therefore, $n = 11$. (The polygon has 11 sides.)

1997-401 c2 v4

MATH FACTOR

PERMUTATIONS AND COMBINATIONS



Lesson 4:
Probability

Student Workbook

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This guide accompanies **Probability** from the Math Factor series.

Writer: Gordon Mallett
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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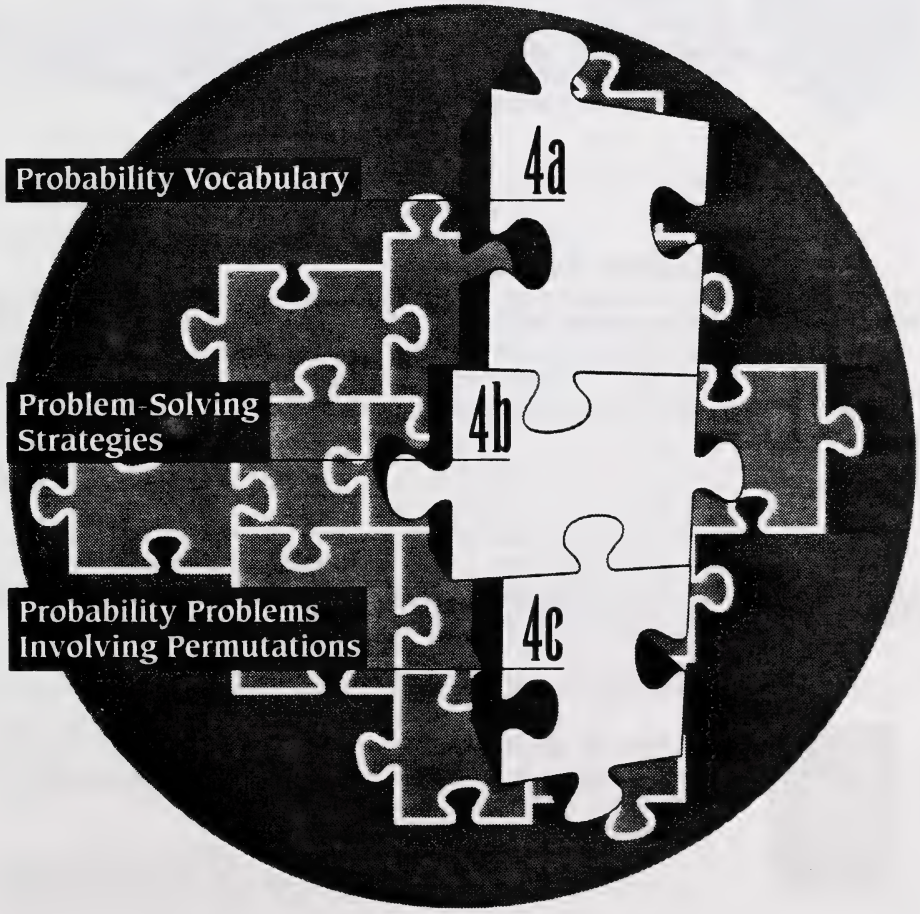
Lesson 4: PROBABILITY

LEARNER XPECTATIONS

By the end of sson 4 you should be able to

- demonstrate an ability to solve probability problems that involve the use of permutations
- use and understand basic probability vocabulary
- solve basic probability problems involving coins, dice, cards and Venn diagrams
- apply the multiplication rule for independent probability

his lesson has three parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Probability Vocabulary

Definition: The probability of an event, E, occurring is determined as follows:

$$P(E) = \frac{\text{number of outcomes in the event}}{\text{number of outcomes in the sample space}}$$

It is important that you understand the meaning of the following terms when working with problems involving probability. Watch for these terms in Video Segment 4a.

- probability
- sample space
- event
- subset
- outcome
- equally likely events
- certain event
- impossible event
- experimental probability
- theoretical probability

MATH FACTOR

Watch the Introduction and Video Segment 4a.

Example

Match each expression in List B with the term in List A.

List A

____ 1. probability
____ 2. sample space
____ 3. event
____ 4. subset
____ 5. outcome
____ 6. equally likely events
____ 7. certain event
____ 8. impossible event
____ 9. experimental probability
____ 10. theoretical probability

List B

a. probability determined by logical deduction
b. a set whose members each belong to a larger set
c. an event E for which $P(E) = 0$
d. any subset of the sample space
e. events E_1 and E_2 for which $P(E_1) = P(E_2)$
f. a collection of all possible outcomes
g. probability determined by doing an experiment
h. a number between 0 and 1 which shows the likelihood of some event happening
i. a single element of a sample space
j. an event E for which $P(E) = 1$

Solution

1h, 2f, 3d, 4b, 5i, 6e, 7j, 8c, 9g, 10a

TEXT

Additional Examples
Mathematics: Principles and Process 12: p. 416
Holtmath 12: p. 363

Exercises

1. A standard deck of 52 cards is thoroughly shuffled. Determine the probability of choosing the following:

   a. a king

   b. a red card

   c. a black ten

2. Each of the letters in **NEW YORK** is written on a separate slip of paper and placed in a hat. One letter is drawn at random. Determine the probability that the letter drawn is the following:

   a. a vowel

   b. a consonant

   c. the same as a letter in the word **MIAMI**

EXERCISE 4a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics: Principles and Process 12: p. 417 (Questions 1-3)
Holtmath 12: p. 364 (Question 1)

Problem-Solving Strategies

Challenge yourself

By using a tree diagram to illustrate all possible outcomes of tossing three coins, find the probability of tossing two heads and a tail in any order.

MATH FACTOR

Watch Video Segment 4b.

Several principles, procedures and understandings are required in order to have an effective strategy for solving probability problems.

- The Fundamental Counting Principle can be written in modified form to apply to probability.

  If the probability of an event occurring is p and the probability of another event occurring is q and so on, then the probability of the events all occurring is $p \times q \times \ldots$.

- Any two events are independent if one event occurring does not affect the probability of the other event occurring. (For example, throwing the number 2 and then the number 5 on two successive throws of a die are independent events.)

  The multiplication rule for independent events A and B is $P(A \text{ and } B) = P(A) \times P(B)$.

- If P(E) represents the probability of an event happening and $P(\overline{E})$ represents the probability of the event not happening, then $P(E) + P(\overline{E}) = 1$.

Example 1

Two dice are rolled. What is the probability that they total 11?

Solution

Method 1

The answer can be obtained using a six-by-six chart like the one used in the first problem in Video Segment 4b. In the chart, only two cells contain the total 11.

$$\therefore \text{ probability } = \frac{2}{36}$$
$$= \frac{1}{18}$$

Method 2

Only the independent events of rolling a 5 on one die and a 6 on the other die total 11.

$$P(5 \text{ on die 1 and 6 on die 2}) = P(5 \text{ on die 1}) \times P(6 \text{ on die 2})$$
$$= \frac{1}{6} \times \frac{1}{6}$$
$$= \frac{1}{36}$$

$$P(6 \text{ on die 1 and 5 on die 2}) = P(6 \text{ on die 1}) \times P(5 \text{ on die 2})$$
$$= \frac{1}{6} \times \frac{1}{6}$$
$$= \frac{1}{36}$$

$$\therefore \text{ overall probability} = \frac{1}{36} + \frac{1}{36} = \frac{1}{18} \doteq 0.056$$

Example 2

In four tosses of a coin, determine the probability of the following:

- four heads occurring
- only one head occurring

Solution

The probability can be calculated by constructing a tree diagram or by applying the multiplication rule.

$$P(\text{four heads}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$
$$= \frac{1}{16}$$

The probability of all four tosses resulting in heads is $\frac{1}{16}$.

$$P(\text{one head and three tails}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$
$$= \frac{1}{16}$$

But the one head could also result on any of the second, third or fourth tosses.

$$\therefore \text{ overall probability } = 4\left(\frac{1}{16}\right)$$
$$= \frac{1}{4}$$

The probability of only one head occurring is $\frac{1}{4}$.

TEXT

Additional Examples
Mathematics: Principles and Process 12: p. 420
Holtmath 12: pp. 368, 370

Exercises

3. L, M and N are independent events. $P(L) = \frac{2}{3}$, $P(M) = \frac{1}{4}$ and $P(N) = \frac{1}{3}$. Find the probability of the following:

   a. L and M will occur

   b. L and M will occur, but N will not occur

4. One die is rolled twice. Determine the probability of the following:

   a. the sum of the numbers is three or more

   b. the difference of the numbers is five

5. Five family members of noticeably different heights line up at random for a family photo. What is the probability that they line up in descending order of height, left to right?

6. In a poll, 22% of symphony subscribers said they liked the music of neither Mozart nor Bach. Altogether, 59% liked Mozart and 40% liked Bach. Use a Venn diagram to determine the probability that a subscriber chosen at random likes both Mozart and Bach.

EXERCISE 4b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 504 (Questions 1-3, 5, 8)
Mathematics: Principles and Process 12: p. 417 (Questions 5, 6), p. 410 (Questions 4, 5)
Holtmath 12: p. 364 (Question 2), p. 369 (Questions 1, 2), p. 371 (Questions 1, 2)

Probability Problems Involving Permutations

Challenge yourself

Carl, Jason, Nelson and four others are in a race. If all are equally skilled, what is the probability that Carl, Jason and Nelson finish first, second and third respectively?

MATH FACTOR

Watch Video Segment 4c.

In addition to adding permutation methods to your strategy for solving probability problems, it is useful to be familiar with **dependent** events.

Any two events are dependent if the occurrence of a first event affects the probability that the other event will occur. For example, medical researchers have proven that smoking cigarettes and getting heart disease are dependent events.

The multiplication rule for dependent events A and B is

$P(\text{A and B}) = P(\text{A}) \times P(\text{B}|\text{A})$.

Note: $P(\text{B}|\text{A})$ means the probability of B, given that event A has already occurred.

Knowledge of dependent events can be used as an alternate method for solving one of the problems presented in Video Segment 4c.

Example 1

Carl, Jason, Nelson and four other equally skilled cyclists are entered in a race. Find the probability that Carl, Jason and Nelson finish first, second and third respectively.

(In Video Segment 4c, the permutation solution was $\frac{1(4!)}{7!} = \frac{24}{5040} \doteq 0.004\,76$.)

Solution

P(Carl finishing first) = $\frac{1}{7}$

Given that Carl has already finished first, P(Jason finishing second) = $\frac{1}{6}$.

Similarly, $P(\text{Nelson finishing third}) = \frac{1}{5}$.

So, altogether the probability is $\frac{1}{7} \times \frac{1}{6} \times \frac{1}{5} = \frac{1}{210} \doteq 0.004\,76$.

Example 2

Permutation methods are used in the following examples:

A four-digit number code is made by randomly assigning the digits 1, 2, 3, 4 and 5 with no repetitions. What is the probability that the code is the number 1234?

Use the probability definition.

$$\text{probability} = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

(**Note:** Favourable outcomes are those outcomes for which the probability is being calculated.)

Solution

There are ${}_5P_4$ possible arrangements of the digits.

$$\begin{aligned} {}_5P_4 &= \frac{5!}{(5-4)!} \\ &= 5! \\ &= 120 \end{aligned}$$

Only one of these arrangements, 1234, is favourable.

$$\begin{aligned} \text{probability} &= \frac{\text{number of favourable cases}}{\text{number of possible cases}} \\ &= \frac{1}{{}_5P_4} \\ &= \frac{1}{120} \end{aligned}$$

Example 3

In a sales promotion, a company offers a cash prize to anyone who can correctly pick the order in which they have randomly arranged the letters of the word **SUPERSAVINGS**. What is the probability that a pick will be the winner?

Solution

To find the number of possible arrangements, apply the permutations rule for repetitions, $\frac{n!}{a!b!c!\ldots}$. There are 12 letters, so $n = 12$. S occurs three times, so $a = 3$.

There is a total of $\frac{12!}{3!} = 79\,833\,600$ arrangements.

Any pick is only one amongst all the possible arrangements. So, the probability that a pick wins is $\frac{1}{79\,833\,600}$.

TEXT

Additional Examples
Mathematics 12: p. 501 (Example 1, *Using permutations*), p. 502 (Example 2a)
Mathematics: Principles and Process 12: p. 418

Exercises

7. In Example 3, what is the probability that a single pick is a winner if there is a restriction that all the vowels must occur first?

8. There are 2 cards drawn, one after another at random, from a standard deck of 52 cards. Use permutations to determine the probability that the first card is a diamond and the second card is a black ace.

9. Solve question 8 by applying the multiplication rule for dependent events.

EXERCISE 4c SOLUTIONS are at the back of this workbook.

TEXT

Additional Examples
Mathematics 12: p. 504 (Question 9)
Mathematics: Principles and Process 12: p. 419 (Questions 2, 5)
Holtmath 12: p. 364 (Question 3)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Go back to the video and study those parts that you find difficult. You must be familiar with methods in this lesson before continuing on to **Lesson 5: Probability: Combinatorics**.

Solutions

EXERCISE 4a

1. a. There are 52 equally likely outcomes in the sample space. The event, K, of choosing a king has four outcomes: K♣, K♦, K♥ and K♠.

$$P(K) = \frac{\text{number of elements in the event K}}{\text{number of elements in the sample space}}$$
$$= \frac{4}{52}$$
$$= \frac{1}{13}$$

b. The event, R, of choosing a red card has $\frac{52}{2} = 26$ outcomes.

$$P(R) = \frac{26}{52}$$
$$= \frac{1}{2}$$

c. The event, T, of choosing a black ten has two outcomes: 10♣, 10♠.

$$P(T) = \frac{2}{52}$$
$$= \frac{1}{26}$$

2. a. There are seven equally likely outcomes in the sample space: {N, E, W, Y, O, R, K}. The event, V, of choosing a vowel has two outcomes: {E, O}.

$$P(V) = \frac{2}{7}$$

b. The event, C, of choosing a consonant has five outcomes: {N, W, Y, R, K}.

$$P(C) = \frac{5}{7}$$

An alternate solution is $P(C) = 1 - P(V)$

$$= 1 - \frac{2}{7}$$
$$= \frac{5}{7}$$

c. None of the letters in **MIAMI** are also in **NEW YORK.**

$$\therefore \text{ probability} = \frac{0}{7}$$
$$= 0$$

EXERCISE 4b

3. a. $P(L \text{ and } M) = P(L) \times P(M)$

$$= \frac{2}{3} \times \frac{1}{4}$$
$$= \frac{1}{6}$$

b. $P(\overline{N}) = 1 - P(N)$

$$= 1 - \frac{1}{3}$$
$$= \frac{2}{3}$$

$$P(L \text{ and } M \text{ and } \overline{N}) = P(L) \times P(M) \times P(\overline{N})$$
$$= \frac{2}{3} \times \frac{1}{4} \times \frac{2}{3}$$
$$= \frac{1}{9}$$

4. a. Only one sum, two, is less than three. It occurs only once, when each roll results in the number 1. The probability of this occurring is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

$$P(\text{a sum of 3 or more}) = 1 - P(\text{a sum less th}$$
$$= 1 - \frac{1}{36}$$
$$= \frac{35}{36}$$

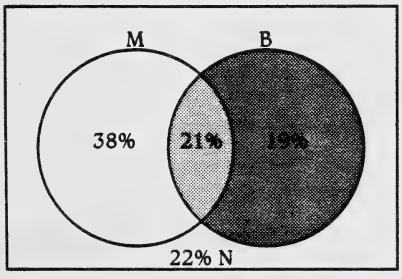
b. To get a difference of five, the first roll must be six and the second must be one, or vice versa.

$$\therefore P = \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6}$$
$$= \frac{1}{36} \times \frac{1}{36}$$
$$= \frac{1}{18}$$

5. The five family members can line up in 5! different ways. Only one of these is the required descending order.

$$\therefore \text{probability} = \frac{1}{5!}$$
$$= \frac{1}{120}$$

6. 21%



EXERCISE 4c

7. There are four different vowels. They can be arranged in ${}_4P_4 = 4! = 24$ ways. Three of the eight consonants are S. So, the consonants can be arranged in $\frac{8!}{3!} = 6720$ ways.
Altogether, there are $24 \times 6720 = 161\,280$ possible arrangements.

A single pick is only one amongst all the possible arrangements.

$$\therefore \text{probability of winning} = \frac{1}{161\,280}$$

8. There are ${}_{52}P_2$ possible two-card arrangements representing the results of the two draws. There are $13 \times 2 = 26$ ways to choose one of 13 diamonds and one of 2 black aces (A♣, A♠).

$$\therefore \text{probability} = \frac{26}{{}_{52}P_2}$$
$$= \frac{26}{2652}$$
$$= \frac{1}{102}$$

9. Let D be the event of drawing a diamond.
Let A be the event of drawing the A♣ or A♠.

$$P(\text{D and A}) = P(\text{D}) \times P(\text{A}|\text{D})$$
$$= \frac{13}{52} \times \frac{2}{51}$$
$$= \frac{26}{2652}$$
$$= \frac{1}{102}$$

Only 51 cards remain after the first draw is complete.

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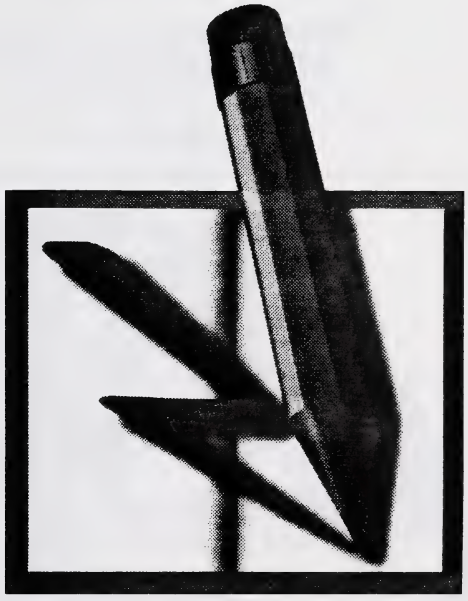
MATH FACTOR

PERMUTATIONS AND COMBINATIONS



Lesson 5: Probability: Combinatorics

Student Workbook

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Alberta EDUCATION

This guide accompanies **Probability: Combinatorics** from the Math Factor series.

Writer: Marilyn Shannon
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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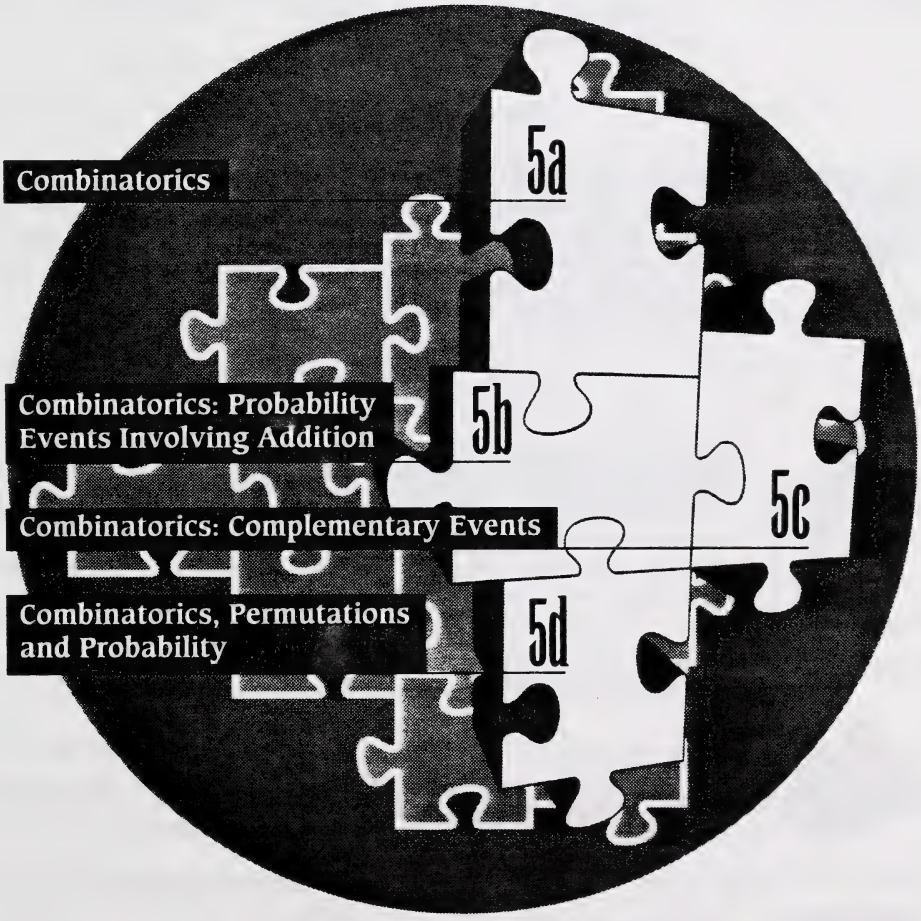
Lesson 5: PROBABILITY: COMBINATORICS

LEARNER PECTATIONS

By the end of n 5 you should be able to

- solve probability problems that involve the use of combinations

s lesson has four parts:



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Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

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Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Combinatorics

What is the probability that a 13-card bridge hand will consist entirely of clubs?

Watch the Introduction and Video Segment 5a.

Recall the following:

- Probability of an event = P(E)

$$P(E) = \frac{\text{number of ways an event can occur}}{\text{total number of possible outcomes}}$$

The probability an event will occur is $0 \le P(E) \le 1$.
If $P(E) = 0$, then the event will not occur.
If $P(E) = 1$, then the event will occur.

- If n objects are chosen r at a time, then the number of combinations is as follows:

$$_nC_r = \frac{n!}{(n-r)!r!} \text{ or } \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Probability questions that involve **choosing** people or objects, card hands or test groups where **order of selection does not matter** are usually done by using combinations or combinatorics.

Example 1

A jar contains six black marbles, four red marbles and two green marbles. If three marbles are drawn at random, what is the probability they are all black?

Solution

There are 6 black marbles, or $_6C_3$ ways of selecting 3 black marbles.
There are 12 marbles in total, or $_{12}C_3$ ways of selecting 3 marbles.

$$\begin{aligned} P(\text{all black marbles}) &= \frac{_6C_3}{_{12}C_3} \\ &= \frac{20}{220} \\ &= \frac{1}{11} \\ &\doteq 0.091 \end{aligned}$$

The probability of choosing 3 black marbles is $\frac{1}{11}$ or approximately 9.1%.

Example 2

There are 12 boys and 8 girls in a class. If a group of 5 students is chosen at random to play a game, find the probability that two of them are girls.

Solution

If two girls are chosen, then there must be three boys chosen.

$$P(\text{2 girls, 3 boys}) = \frac{{}_8C_2 \times {}_{12}C_3}{{}_{20}C_5}$$

Choose 2 of 8 girls, Choose 3 of 12 boys / Choose 5 of 20 students

$$= \frac{28 \times 220}{15\,504}$$

$$= \frac{385}{969}$$

$$\doteq 0.397$$

The probability of choosing two girls is $\frac{385}{969}$ or approximately 39.7%.

TEXT

Additional Examples
Mathematics 12: p. 502 (Examples 2b, 3)
Mathematics: Principles and Process 12: p. 418

Exercises

1. Three toy cars are chosen at random from nine toy cars, where four are defective. What is the probability that one of the chosen cars is defective?

2. Five cards are drawn at random from a standard deck of 52 cards. Find the probability of the following:

   a. all five cards are hearts

   b. there are two hearts and three spades

3. Eleven cards are numbered from 1 to 11 and put into a box. If two cards are drawn at random, what is the probability that both cards drawn are odd numbers?

4. If the occurrence of both sexes is equally likely, calculate the following probabilities for a family of four children:

   a. the children are all boys

   b. there are three girls and one boy

5. There are 8 males in a group of 20 people, where 3 people are chosen for a committee. What is the probability that the 3 people chosen are the following?

   a. all male

   b. all female

   c. 2 males and 1 female

EXERCISE 5a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 504 (Questions 6, 8-10, 12, 13)
Mathematics: Principles and Process 12: p. 417 (Question 5),
p. 419 (Questions 1a, 1b, 2a, 2b, 6)
Holtmath 12: p. 371 (Questions 3, 10a)

Combinatorics: Probability Events Involving Addition

An insurance salesperson is reviewing 8 policies that are randomly selected from a group of 10 life insurance policies, 7 car insurance policies and 14 homeowner policies. Calculate the probability that the salesperson will select, at most, 1 life insurance policy.

The words **at least** and **at most** in a problem imply the use of addition.

Example 1

A study group contains 22 right-handers and 6 left-handers. If a sample of three is chosen from the group, what is the probability that the sample contains at least 1 left-hander?

Solution

There are three different situations or cases, any of which can occur to satisfy the conditions in the question (i.e., one left-hander, two left-handers or three left-handers).

$$\begin{aligned} P(\text{at least one left-hander}) &= P(1 \text{ left-hander}) + P(2 \text{ left-handers}) + P(3 \text{ left-handers}) \\ &= \frac{{}_6C_1 \times {}_{22}C_2}{{}_{28}C_3} + \frac{{}_6C_2 \times {}_{22}C_1}{{}_{28}C_3} + \frac{{}_6C_3}{{}_{28}C_3} \\ &= \frac{1386}{3276} + \frac{330}{3276} + \frac{20}{3276} \\ &\doteq 0.4231 + 0.1007 + 0.0061 \\ &\doteq 0.53 \end{aligned}$$

Thus, the probability of the sample containing at least one left-hander is approximately 0.53 or about 53%.

Example 2

Using the information given in Example 1, what is the probability that there is at most one right-hander?

Solution

There are two different cases that satisfy the conditions: no right-handers and one right-hander.

$$\begin{aligned} P(\text{at most 1 right-hander}) &= P(0 \text{ right-handers}) + P(1 \text{ right-hander}) \\ &= \frac{{}_6C_3}{{}_{28}C_3} + \frac{{}_6C_2 \times {}_{22}C_1}{{}_{28}C_3} \\ &= \frac{20}{3276} + \frac{330}{3276} \\ &\doteq 0.0061 + 0.1007 \\ &\doteq 0.1068 \end{aligned}$$

Thus, the probability of the sample containing at most one right-hander is approximately 0.1068 or about 11%.

Exercises

6. There are 15 boys and 10 girls in a class. A yearbook committee of four is chosen from the class at random.

   a. What is the probability that at least 1 girl is chosen?

   b. What is the probability that at most there will be 2 boys on the committee?

7. There are ten mathematics books and eight chemistry books. If five books are chosen from the mathematics and chemistry books, what is the probability that at least four will be mathematics books?

8. Three coins are tossed. What is the probability of at least two landing heads up?

9. If 5 cards are drawn at random from a standard deck of 52 cards, then what is the probability that there are at most 2 black cards drawn?

10. There are six black marbles, five green marbles and four yellow marbles in a bag. If three marbles are drawn at random from the bag, what is the probability that all the marbles are the same colour?

EXERCISE 5b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 505 (Questions 17, 19)
Holtmath 12: p. 369 (Question 8), p. 371 (Questions 4, 10)

Combinatorics: Complementary Events

Challenge yourself

Four married couples are on a committee. If a man and a woman are chosen at random to lead the committee, what is the probability that they are not husband and wife?

MATH FACTOR

Watch Video Segment 5c.

If the probability of an event occurring is P(E) and the probability the event will not occur is $P(\overline{E})$, then $P(E)+P(\overline{E})=1$ or $P(\overline{E})=1-P(E)$.

For example, if a multiple-choice question has four responses, the probability of guessing the correct answer is $\frac{1}{4}$ and the probability of not guessing the correct answer is $1-\frac{1}{4}=\frac{3}{4}$. Using the idea of complementary events is useful in many of the problems that involve addition.

Example 1

A coach randomly chooses a basketball team of five players from a group of 15 girls. If 3 of the girls are sisters, what is the probability that at least one of the sisters will be on the team?

Solution

This problem can be done two ways: using addition or using complementary events.

Method 1
As you saw in 5b, the probability of at least one sister can be determined by:
$P(\text{at least 1 sister}) = P(\text{1 sister on the team}) + P(\text{2 sisters}) + P(\text{3 sisters})$

Method 2
$P(\text{at least 1 sister}) = 1 - P(\text{0 sisters on the team})$

$$= 1 - \frac{{}_{12}C_5}{{}_{15}C_5}$$
$$= 1 - \frac{792}{3003}$$
$$= 1 - \frac{24}{91}$$
$$= \frac{67}{91}$$
$$\doteq 0.74$$

Thus, the probability that at least one sister will be on the team is approximately 0.74 or about 74%.

Example 2

A committee of eight is to be chosen at random from 15 French-speaking Canadians and 10 English-speaking Canadians. What is the probability there will be at most 6 French-speaking Canadians on the committee?

Solution

Again, this problem can be done in two ways:

Method 1

$$P(\text{at most 6 French}) = P(0 \text{ French}) + P(1 \text{ French}) + \ldots + P(5 \text{ French}) + P(6 \text{ French})$$

Method 2

$$\begin{aligned} P(\text{at most 6 French}) &= 1 - \left[P(7 \text{ French}) + P(8 \text{ French}) \right] \\ &= 1 - \left[\frac{({}_{15}C_7)({}_{10}C_1)}{{}_{25}C_8} + \frac{{}_{15}C_8}{{}_{25}C_8} \right] \\ &= 1 - \left[\frac{64\,350}{1\,081\,575} + \frac{6435}{1\,081\,575} \right] \\ &= 1 - \left[\frac{70\,785}{1\,081\,575} \right] \\ &= \frac{1\,010\,790}{1\,081\,575} \\ &\doteq 0.9346 \end{aligned}$$

Thus, the probability of at most six French-speaking Canadians being on the committee is approximately 0.9346 or about 93%.

Exercises

11. A jar contains six red balls, eight white balls and four blue balls. If three balls are drawn at random, determine the probability of the following:

    a. all the balls are different colours

    b. two or more of the balls are the same colour

12. If the occurrence of both sexes are equally likely for a family that has four children, then find the probability of the following:

    a. all the children are girls

    b. there is at least one boy

13. There were 100 tickets sold for a dance and three door prizes were awarded. If the Tran family had 4 tickets, what is the probability that they won at least one prize?

14. A coach selects a team of 8 players from a group of 20 by drawing names from a hat. Twins, Jeff and Jason, are among the 20 players. Determine the probability of the following:

    a. both twins are picked

    b. neither twin is picked

    c. at least one twin is picked

15. If 13 cards are randomly selected from a standard deck of 52 cards, then what is the probability that at most 10 will be red?

EXERCISE 5c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 505 (Questions 16b, 17)
Holtmath 12: p. 371 (Questions 4, 8)

Combinatorics, Permutations and Probability

Challenge yourself

What is the probability that a blackout will occur on a MATHO card if 74 numbers have been called?

Watch Video Segment 5d.

Often, you can calculate probabilities using either permutations or combinations and sometimes the Fundamental Counting Principle. The method you use depends on how you interpret the problem.

Example

If 3 cards are picked at random from a deck of 52 regular playing cards, what is the probability that they are all kings?

Solution

Method 1: Using Permutations

$$P(\text{3 kings}) = \frac{{}_4P_3}{{}_{52}P_3}$$

$$= \frac{\frac{4!}{1!}}{\frac{52!}{49!}}$$

$$= \frac{24}{132\,600}$$

$$\doteq 0.000\,18$$

Using the Fundamental Counting Principle, the first king can be selected in four ways, the second king in three ways and the third king in two ways; or there are ${}_4P_3$ ways.

Method 2: Using Combinations

$$P(\text{3 kings}) = \frac{{}_4C_3}{{}_{52}C_3}$$

$$= \frac{\frac{4!}{(1!3!)}}{\frac{52!}{(49!3!)}}$$

$$= \frac{24}{132\,600}$$

$$\doteq 0.000\,18$$

The number of ways of choosing 3 kings from 4 kings is ${}_4C_3$. The number of ways of choosing 3 cards from 52 is ${}_{52}C_3$.

Notice that the 3! is a factor of both the numerator and the denominator, resulting in the same calculations as used in Method 1.

Method 3: Using the Fundamental Counting Principle

$$P(3 \text{ kings}) = \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50}$$
$$= \frac{24}{132\,600}$$
$$\doteq 0.000\,18$$

On the first draw, there are four choices for a king out of 52 cards. If a king is drawn on the first draw, then there are three choices on the second draw from only 51 cards. On the third draw, there are two kings and 50 cards left.

Thus, the probability of drawing three kings is $\frac{24}{132\,600}$ or approximately 0.000 18.

TEXT

Additional Examples
Mathematics 12: p. 501

Exercises

16. From an ordinary deck of 52 cards, 4 cards are drawn at random. What is the probability that all 4 cards are black?

17. From 20 members of a club, there is a random selection to fill three executive positions (president, treasurer and secretary).

    a. What is the probability that Sue, George and Harry will be selected?

    b. What is the probability that Sue will be president, George will be treasurer and Harry will be secretary?

18. A slowpitch team has ten male members and eight female members. If two members are selected at random, find the probability of the following:

    a. both are female

b. both are male

c. one is male and the other is female

19. In a Lotto 5/48, five numbers are drawn at random from the numbers 1 to 48. If Tom bought a ticket, determine the probability of the following:

a. all five numbers Tom chose were drawn

b. four of Tom's numbers were drawn

c. three of Tom's numbers were drawn

d. none of Tom's numbers were drawn

EXERCISE 5d SOLUTIONS are at the back of this workbook.

TEXT

Additional Examples
Mathematics 12: p. 504 (Questions 1-4, 7, 11, 14, 15)
Mathematics: Principles and Process 12: p. 417 (Questions 2, 3),
p. 419 (Questions 1c, 2c, 3, 4, 8)
Holtmath 12: p. 364 (Questions 1-4)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check to see if you have mastered the expectation identified. Solving probability problems using combinatorics is a very useful skill. The expansion of binomials and how they are related to Pascal's Triangle is explored in **Lesson 6: Binomial Theorem (Pascal's Triangle)**.

Solutions

EXERCISE 5a

1. $P(\text{1 defective}) = \dfrac{{}_4C_1 \times {}_5C_2}{{}_9C_3}$

$= \dfrac{4 \times 10}{84}$

$= \dfrac{40}{84}$

$= \dfrac{10}{21}$

2. a. $P(\text{all hearts}) = \dfrac{{}_{13}C_5}{{}_{52}C_5}$

$= \dfrac{1287}{2\,598\,960}$

$= \dfrac{99}{199\,920}$

$\doteq 0.0005$

b. $P(\text{2 hearts, 3 spades}) = \dfrac{{}_{13}C_2 \times {}_{13}C_3}{{}_{52}C_5}$

$= \dfrac{78 \times 286}{2\,598\,960}$

$= \dfrac{143}{16\,660}$

$\doteq 0.0086$

3. $P(\text{both odd}) = \dfrac{{}_6C_2}{{}_{11}C_2}$

$= \dfrac{15}{55}$

$= \dfrac{3}{11}$

4. a. $P(\text{all boys}) = \dfrac{{}_4C_4}{(2)^4}$

$= \dfrac{1}{16}$

b. $P(\text{3 girls, 1 boy}) = \dfrac{{}_4C_3 \times {}_1C_1}{(2)^4}$

$= \dfrac{4}{16}$

$= \dfrac{1}{4}$

5. a. $P(\text{all male}) = \dfrac{{}_8C_3}{{}_{20}C_3}$

$= \dfrac{56}{1140}$

$= \dfrac{14}{285}$

b. $P(\text{all female}) = \dfrac{{}_{12}C_3}{{}_{20}C_3}$

$= \dfrac{220}{1140}$

$= \dfrac{11}{57}$

c. $P(\text{2 males, 1 female}) = \dfrac{{}_8C_2 \times {}_{12}C_1}{{}_{20}C_3}$

$= \dfrac{336}{1140}$

$= \dfrac{28}{95}$

EXERCISE 5b

6. a. $P(\text{at least 1 girl}) = P(\text{1 girl}) + P(\text{2 girls}) + P(\text{3 girls}) + P(\text{4 girls})$

$= \dfrac{{}_{10}C_1 \times {}_{15}C_3}{{}_{25}C_4} + \dfrac{{}_{10}C_2 \times {}_{15}C_2}{{}_{25}C_4} + \dfrac{{}_{10}C_3 \times {}_{15}C_1}{{}_{25}C_4} + \dfrac{{}_{10}C_4}{{}_{25}C_4}$

$= \dfrac{4500}{12\,650} + \dfrac{4725}{12\,650} + \dfrac{1800}{12\,650} + \dfrac{210}{12\,650}$

$= \dfrac{11\,285}{12\,650}$

$= \dfrac{2257}{2530}$

$\doteq 0.892$

b. $P(\text{at most 2 boys}) = P(\text{0 boys}) + P(\text{1 boy}) + P(\text{2 boys})$

$= \dfrac{{}_{10}C_4}{{}_{25}C_4} + \dfrac{{}_{15}C_1 \times {}_{10}C_3}{{}_{25}C_4} + \dfrac{{}_{15}C_2 \times {}_{10}C_2}{{}_{25}C_4}$

$= \dfrac{210}{12\,650} + \dfrac{1800}{12\,650} + \dfrac{4725}{12\,650}$

$= \dfrac{6735}{12\,650}$

$= \dfrac{1347}{2530}$

$\doteq 0.532$

7. $P(\text{at least 4 math}) = P(\text{4 math}) + P(\text{all math})$

$= \dfrac{{}_{10}C_4 \times {}_8C_1}{{}_{18}C_5} + \dfrac{{}_{10}C_5}{{}_{18}C_5}$

$= \dfrac{1680}{8568} + \dfrac{252}{8568}$

$= \dfrac{1932}{8568}$

$= \dfrac{23}{102}$

$\doteq 0.225$

8. $P(\text{at least 2 heads}) = P(2 \text{ heads}) + P(3 \text{ heads})$

$$= \frac{{}_3C_2}{(2)^3} + \frac{{}_3C_3}{(2)^3}$$
$$= \frac{3}{8} + \frac{1}{8}$$
$$= \frac{1}{2}$$

9. $P(\text{at most 2 black cards}) = P(0 \text{ blacks}) + P(1 \text{ black}) + P(2 \text{ blacks})$

$$= \frac{{}_{26}C_5}{{}_{52}C_5} + \frac{{}_{26}C_1 \times {}_{26}C_4}{{}_{52}C_5} + \frac{{}_{26}C_2 \times {}_{26}C_3}{{}_{52}C_5}$$
$$\doteq 0.025\,31 + 0.149\,56 + 0.325\,13$$
$$\doteq 0.5$$

10. $P(\text{all the same colour}) = P(\text{all black}) + P(\text{all green}) + P(\text{all yellow})$

$$= \frac{{}_6C_3}{{}_{15}C_3} + \frac{{}_5C_3}{{}_{15}C_3} + \frac{{}_4C_3}{{}_{15}C_3}$$
$$= \frac{20}{455} + \frac{10}{455} + \frac{4}{455}$$
$$= \frac{34}{455}$$
$$\doteq 0.075$$

EXERCISE 5c

11. a. $P(\text{all different}) = P(1 \text{ red}, 1 \text{ white}, 1 \text{ blue})$

$$= \frac{{}_6C_1 \times {}_8C_1 \times {}_4C_1}{{}_{18}C_3}$$
$$= \frac{192}{816}$$
$$= \frac{4}{17}$$

b. $P(\text{2 or more same colour}) = 1 - P(\text{all different})$

$$= 1 - \frac{4}{17}$$
$$= \frac{13}{17}$$

12. a. $P(\text{all girls}) = \frac{{}_4C_4}{(2)^4}$

$$= \frac{1}{16}$$

b. $P(\text{at least 1 boy}) = 1 - P(\text{all girls})$

$$= 1 - \frac{1}{16}$$
$$= \frac{15}{16}$$

13. $P(\text{won at least 1 prize}) = 1 - P(\text{not winning})$

$$= 1 - \frac{{}_{96}C_3}{{}_{100}C_3}$$
$$\doteq 1 - 0.8836$$
$$\doteq 0.1164$$

14. a. $P(\text{both}) = \dfrac{{}_2C_2 \times {}_{18}C_6}{{}_{20}C_8}$

$$= \frac{18\,564}{125\,970}$$
$$\doteq 0.1474$$

b. $P(\text{neither}) = \dfrac{{}_{18}C_8}{{}_{20}C_8}$

$$= \frac{43\,758}{125\,970}$$
$$\doteq 0.3474$$

c. $P(\text{at least 1}) = 1 - P(\text{neither})$

$$\doteq 1 - 0.3474$$
$$\doteq 0.6526$$

15. $P(\text{at most 10 red}) = 1 - \left[P(11\text{ red}) + P(12\text{ red}) + P(13\text{ red})\right]$

$$= 1 - \left[\frac{{}_{26}C_{11} \times {}_{26}C_2}{{}_{52}C_{13}} + \frac{{}_{26}C_{12} \times {}_{26}C_1}{{}_{52}C_{13}} + \frac{{}_{26}C_{13}}{{}_{52}C_{13}}\right]$$
$$= 1 - \left[0.003\,954 + 0.000\,395 + 0.000\,016\right]$$
$$= 1 - 0.004\,365$$
$$= 0.995\,635$$

EXERCISE 5d

16. $P(4\text{ blacks}) = \dfrac{{}_{26}C_4}{{}_{52}C_4}$ or $P(4\text{ blacks}) = \dfrac{{}_{26}P_4}{{}_{52}P_4}$

$$= 0.0552 \qquad = 0.0552$$

17. a. $P(\text{Sue, George and Harry}) = \dfrac{{}_3C_3}{{}_{20}C_3}$ or $P(\text{Sue, George and Harry}) = \dfrac{{}_3P_3}{{}_{20}P_3}$

$$= 0.000\,877 \qquad = 0.000\,877$$

b. $P(\text{Sue – president, George – treasurer, Harry – secretary}) = \dfrac{1}{{}_{20}P_3}$

$$= 0.000\,146$$

18. a. $P(\text{both female}) = \dfrac{{}_8C_2}{{}_{18}C_2}$ or $P(\text{both female}) = \dfrac{{}_8P_2}{{}_{18}P_2}$

$$= 0.183 \qquad = 0.183$$

b. $P(\text{both male}) = \frac{{}_{10}C_2}{{}_{18}C_2}$ or $P(\text{both male}) = \frac{{}_{10}P_2}{{}_{18}P_2}$

$= 0.294$ $= 0.294$

c. $P(1 \text{ male}, 1 \text{ female}) = \frac{{}_{10}C_1 \times {}_8C_1}{{}_{18}C_2}$

$= 0.5228$

or $P(1 \text{ male}, 1 \text{ female}) = 1 - \left[P(\text{both male}) + P(\text{both female})\right]$

$= 0.5228$

19. a. $P(\text{all drawn}) = \frac{{}_5C_5}{{}_{48}C_5}$

$= 0.000\,000\,584$

b. $P(4 \text{ drawn}) = \frac{{}_5C_4 \times {}_{43}C_1}{{}_{48}C_5}$

$= 0.000\,126$

c. $P(3 \text{ drawn}) = \frac{{}_5C_3 \times {}_{43}C_2}{{}_{48}C_5}$

$= 0.000\,527$

d. $P(\text{none drawn}) = \frac{{}_{43}C_5}{{}_{48}C_5}$

$= 0.5622$

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MATH FACTOR

PERMUTATIONS AND COMBINATIONS

Lesson 6:
Binomial Theorem (Pascal's Triangle)

Student Workbook

Produced by

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for

Alberta EDUCATION

This guide accompanies **Binomial Theorem (Pascal's Triangle)** from the Math Factor series.

Writer: Marilyn Shannon
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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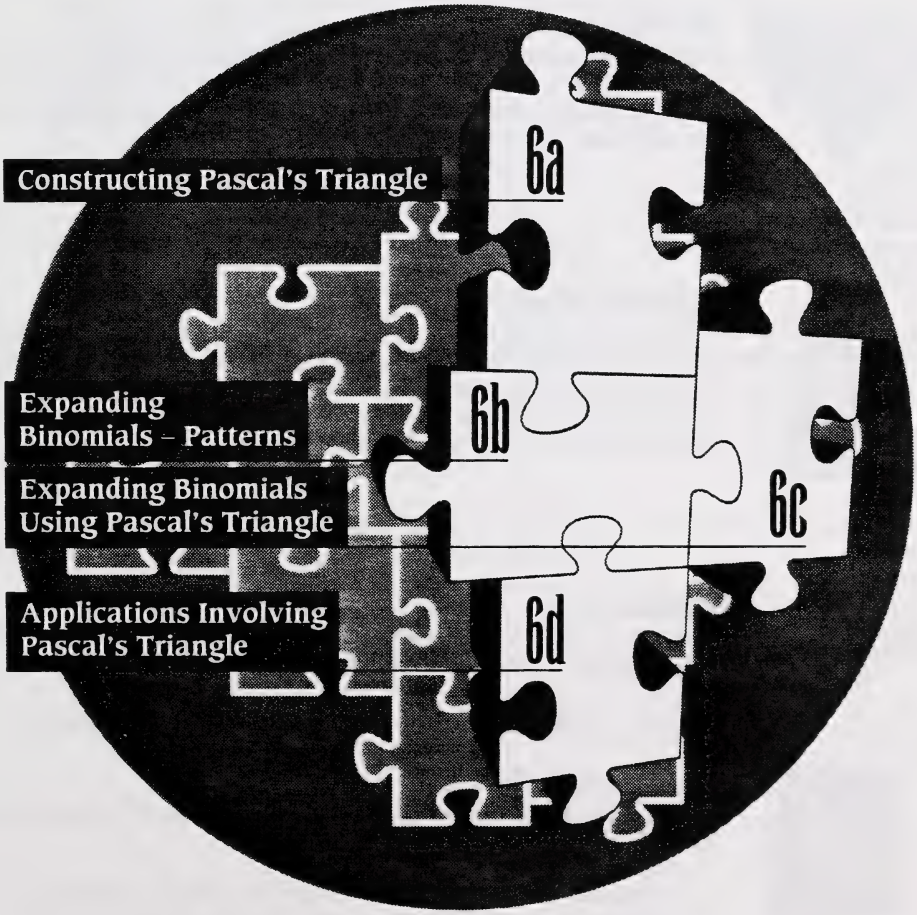
Lesson 6: BINOMIAL THEOREM (PASCAL'S TRIANGLE)

LEARNER PECTATIONS

By the end of on 6 you should be able to

- expand binomials of the form $(x + y)^n$, where n is a whole number, using Pascal's Triangle
- relate the numerical coefficients in a binomial expansion to the terms of Pascal's Triangle, and vice versa

is lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

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This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Constructing Pascal's Triangle

Watch the Introduction and Video Segment 6a.

Additional Examples
Mathematics 12: p. 507
Mathematics: Principles and Process 12: p. 424
Holtmath 12: p. 361

Exercises

1. Write in the missing numbers for the first eight rows of Pascal's Triangle and indicate the number of terms in each row.

Row 1	1	1 term
Row 2	1 1	___ terms
Row 3	1 ___ 1	___ terms
Row 4	1 ___ ___ 1	___ terms
Row 5	1 ___ ___ ___ 1	___ terms
Row 6	1 ___ ___ ___ ___ 1	___ terms
Row 7	1 ___ ___ ___ ___ ___ 1	___ terms
Row 8	1 ___ ___ ___ ___ ___ ___ 1	___ terms

2. Fill in the blanks, using combination notation, to form Pascal's Triangle; then determine the number of terms in each row.

Row 1	${}_0C_0$	___ term
Row 2	${}_1C_0$ ${}_1C_1$	___ terms
Row 3	${}_2C_0$ ____ ${}_2C_2$	___ terms
Row 4	${}_3C_0$ ____ ____ ${}_3C_3$	___ terms
Row 5	${}_4C_0$ ____ ____ ____ ${}_4C_4$	___ terms
Row 6	${}_5C_0$ ____ ____ ____ ____ ${}_5C_5$	___ terms
Row 7	${}_6C_0$ ____ ____ ____ ____ ____ ${}_6C_6$	___ terms
Row 8	${}_7C_0$ ____ ____ ____ ____ ____ ____ ${}_7C_7$	___ terms

3. How are the triangles in questions 1 and 2 related?

4. Determine the number of terms in the following:

   a. 2nd row

   b. 5th row

   c. 8th row

   d. *n*th row

5. Write the 14th row of Pascal's Triangle in combination notation.

EXERCISE 6a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 508 (Question 1)
Holtmath 12: p. 362 (Questions 2, 3)

Expanding Binomials — Patterns

Watch Video Segment 6b.

Additional Examples
Mathematics 12: p. 507
Mathematics: Principles and Process 12: p. 424

Exercises

The following questions are designed to reinforce the patterns that occur when expanding a polynomial.

6. Fill in the blanks by multiplying the following binomials and by constructing the first five rows of Pascal's Triangle.

Expanded Polynomial	Pascal's Triangle
$(x+y)^0 =$ ____	Row 1 ____
$(x+y)^1 =$ ____+____	Row 2 ____ ____
$(x+y)^2 =$ ____+____+____	Row 3 ____ ____ ____
$(x+y)^3 =$ ____+____+____+____	Row 4 ____ ____ ____ ____
$(x+y)^4 =$ ____+____+____+____+____	Row 5 ____ ____ ____ ____ ____

7. Using your answers from question 6, answer the following questions.

   a. How many terms are there in the expansion of:

   i. $(x+y)^3$ iii. $(x+y)^8$ v. $(x+y)^n$

   ii. $(x+y)^0$ iv. $(x+y)^{20}$

   b. The coefficients of the expansion of $(x+y)^2$ are the same as the ____ row of Pascal's Triangle.

   c. The coefficients of the expansion of $(x+y)^4$ are the same as the ____ row of Pascal's Triangle.

d. The coefficients of the expansion of $(x+y)^n$ would be the same as the ___ row of Pascal's Triangle.

e. What is the degree of each term of the expansion of $(x+y)^4$? (Recall that the degree of a term is the exponent the variable is raised to, or that the degree is the sum of the exponents if the term has more than one variable.)

f. What is the degree of each term of the expansion of $(x+y)^n$?

g. Consider the expansion of $(x+y)^3$.
In the first term, x is raised to the ___ power and y is raised to the ___ power.
In the second term, x is raised to the ___ power and y is raised to the ___ power.
In the third term, x is raised to the ___ power and y is raised to the ___ power.
Last, in the fourth term, x is raised to the ___ power and y is raised to the ___ power.

h. Consider the expansion of $(x+y)^n$.
In the first term, x is raised to the ___ power and y is raised to the ___ power.
In the second term, x is raised to the ___ power and y is raised to the ___ power.
In the third term, x is raised to the ___ power and y is raised to the ___ power.
In the last term, x is raised to the ___ power and y is raised to the ___ power.

i. Using the expansion of $(x+y)^7$, complete the following phrases.

- There would be ___ terms.
- The coefficients of the terms would be the same as the ____ row of Pascal's Triangle.
- The first term is _____ and the last term is _____.

j. Expand $(x+y)^7$ without multiplying.

8. Write the missing exponents in each of the following terms of the expansion of $(x+y)^{14}$.

a. ${}_{14}C_{11}x^3y$ b. ${}_{14}C_4x^{10}y$ c. ${}_{14}C_{13}xy$ d. ${}_{14}C_9x\ y^9$

9. Find the number of terms that would be in the expansion of the following:

a. $(x+y)^{22}$ b. $(x+y)^{100}$

10. Write the value of n for which each term is a term in the expansion of $(x+y)^n$.

a. ${}_7C_4x^3y^4$ b. ${}_{22}C_{12}x^{10}y^{12}$ c. ${}_8C_1x^7y$ d. ${}_{16}C_8x^8y^8$

11. Without multiplying, expand the following:

a. $(x+y)^6$

b. $(x+y)^9$

EXERCISE 6b SOLUTIONS are at the back of this workbook.

Summary – Expanding $(x+y)^n$

- The number of terms is $n + 1$.
- The coefficients of the expansion correspond to the $n + 1$ row of Pascal's Triangle. **Hint:** Look for the row in which n occurs.
- In the first term, x is raised to the n exponent and y is raised to the 0 exponent. In each term following, the exponent of x decreases by 1 and the exponent of y increases by 1.
- The degree (exponent or sum of exponents) of each term of the expansion is n.

In general, the expansion looks as follows:

$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + {}_nC_3x^{n-3}y^3 + \ldots + {}_nC_{n-2}x^2y^{n-2} + {}_nC_{n-1}xy^{n-1} + {}_nC_ny^n$, where n is a whole number.

TEXT

Additional Practice
Mathematics 12: p. 508 (Questions 3, 5, 9)
Mathematics: Principles and Process 12: p. 425 (Questions 1a, 3)
Holtmath 12: p. 362 (Questions 9, 11a)

Expanding Binomials Using Pascal's Triangle

Challenge yourself

Expand $(2a+3b)^3$ without multiplying.

MATH FACTOR

Watch Video Segment 6c.

Example

Expand $\left(2a^3-\frac{1}{a}\right)^5$ without multiplying.

Solution

$\left(2a^3-\frac{1}{a}\right)^5$ will expand as $(x+y)^5$. That is, there will be six terms in the expansion and the coefficients of the terms will be the same as the sixth row of Pascal's Triangle.

$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

Substitute $x=2a^3$ and $y=-\frac{1}{a}$ in the previous expansion.

$$\left(2a^3-\frac{1}{a}\right)^5 = \left(2a^3\right)^5+5\left(2a^3\right)^4\left(-\frac{1}{a}\right)+10\left(2a^3\right)^3\left(-\frac{1}{a}\right)^2+10\left(2a^3\right)^2\left(-\frac{1}{a}\right)^3$$
$$+5\left(2a^3\right)\left(-\frac{1}{a}\right)^4+\left(-\frac{1}{a}\right)^5$$
$$= 32a^{15}+5\left(16a^{12}\right)\left(-\frac{1}{a}\right)+10\left(8a^9\right)\left(\frac{1}{a^2}\right)+10\left(4a^6\right)\left(-\frac{1}{a^3}\right)$$
$$+5\left(2a^3\right)\left(\frac{1}{a^4}\right)-\frac{1}{a^5}$$
$$= 32a^{15}-80a^{11}+80a^7-40a^3+\frac{10}{a}-\frac{1}{a^5}$$

Thus, $\left(2a^3-\frac{1}{a}\right)^5 = 32a^{15}-80a^{11}+80a^7-40a^3+\frac{10}{a}-\frac{1}{a^5}$.

TEXT

Additional Example
Mathematics 12: p. 508

Exercises

12. Expand the following binomials without multiplying.

a. $(2p+3q)^5$

b. $(1-m)^6$

c. $(3a^2-2b)^4$

d. $\left(x^2-\frac{2}{x}\right)^3$

13. Expand and simplify $(a+b)^6-(a-b)^6$.

14. a. Calculate $(105)^2$ by writing as $(100+5)^2$.

b. Calculate $(105)^3$ without using a calculator.

EXERCISE 6c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 509 (Questions 6, 11)
Mathematics: Principles and Process 12: p. 425 (Questions 1, 2)
Holtmath 12: p. 362 (Question 11)

Applications Involving Pascal's Triangle

Challenge yourself

How many routes would a bus be able to take from a library to city hall if the bus can only travel north and east and city hall is three blocks north and four blocks east of the library?

MATH FACTOR

Watch Video Segment 6d.

There are several applications of Pascal's Triangle. There are also several interesting patterns that arise from Pascal's Triangle. The following exercises involve patterns and questions that can be solved using Pascal's Triangle. Some of these questions can be solved using other methods. Try them.

Exercises

15. If Alfred's house is six blocks south and three blocks east of school and he only travels south and east, then how many different routes could he take to travel from the school to his house?

16.

	Pascal's Triangle	Sum	2^n
Row 1	1	= _____	= _____
Row 2	1 + 1	= _____	= _____
Row 3	1 + 2 + 1	= _____	= _____
Row 4	1 + 3 + 3 +1	= _____	= _____
Row 5	1 + 4 + 6 + 4 + 1	= _____	= _____
Row 6	1+ 5 + 10 + 10 + 5 + 1	= _____	= _____
Row 7	1 + 6 + 15 + 20 + 15 + 6 + 1	= _____	= _____
Row 8	1 + 7 + 21 + 35 + 35 + 21 + 7 + 1	= _____	= _____

17. Use the binomial expansion of $(x+y)^7$ to prove that
$2^7 = {}_7C_0 + {}_7C_1 + {}_7C_2 + {}_7C_3 + {}_7C_4 + {}_7C_5 + {}_7C_6 + {}_7C_7$.

18. Jane has five stamps: 1¢, 2¢, 5¢, 10¢ and 20¢. How many different sums of postage can be obtained using at least one of these stamps?

19. Determine the number of different positive factors of 910. (**Hint:** Determine the prime factors of 910.)

20. a. Fill in the blanks; then answer the following questions.

Row 1 ${}_0C_0 =$____

Row 2 ${}_1C_0 =$____ ${}_1C_1 =$____

Row 3 ${}_2C_0 =$____ ${}_2C_1 =$____ ${}_2C_2 =$____

Row 4 ${}_3C_0 =$____ ${}_3C_1 =$____ ${}_3C_2 =$____ ${}_3C_3 =$____

Row 5 ${}_4C_0 =$____ ${}_4C_1 =$____ ${}_4C_2 =$____ ${}_4C_3 =$____ ${}_4C_4 =$____

Row 6 ${}_5C_0 =$____ ${}_5C_1 =$____ ${}_5C_2 =$____ ${}_5C_3 =$____ ${}_5C_4 =$____ ${}_5C_5 =$____

Row 7 ${}_6C_0 =$____ ${}_6C_1 =$____ ${}_6C_2 =$____ ${}_6C_3 =$____ ${}_6C_4 =$____ ${}_6C_5 =$____ ${}_6C_6 =$____

b. What is the numerical value of ${}_4C_1$? of ${}_4C_3$?

c. What is the equivalent expression of ${}_6C_4$ in the form ${}_nC_r$? of ${}_5C_2$?

d. What expression would be equivalent to ${}_{10}C_3$ in combination notation? in ${}_nC_r$?

e. Prove that ${}_nC_r = {}_nC_{n-r}$.

f. What is the sum of ${}_3C_1 + {}_3C_2$?

g. What is ${}_4C_2$ equal to?

h. What is the sum of ${}_5C_2 + {}_5C_3$ in combination notation?

i. What is the sum of ${}_6C_3 + {}_6C_4$ in combination notation?

j. What is the sum of ${}_nC_r + {}_nC_{r+1}$ in combination notation?

k. Prove that ${}_{n+1}C_{r+1} = {}_nC_r + {}_nC_{r+1}$.

EXERCISE 6d SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics: Principles and Process 12: p. 425 (Question 7)
Holtmath 12: p. 362 (Questions 4, 5)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Expanding binomials using Pascal's Triangle is a skill needed in order to develop the Binomial Theorem that is presented in **Lesson 7: Binomial Theorem.**

Solutions

EXERCISE 6a

1.

Row 1	1	1 term
Row 2	1 1	2 terms
Row 3	1 2 1	3 terms
Row 4	1 3 3 1	4 terms
Row 5	1 4 6 4 1	5 terms
Row 6	1 5 10 10 5 1	6 terms
Row 7	1 6 15 20 15 6 1	7 terms
Row 8	1 7 21 35 35 21 7 1	8 terms

2.

Row 1	${}_0C_0$	1 term
Row 2	${}_1C_0$ ${}_1C_1$	2 terms
Row 3	${}_2C_0$ ${}_2C_1$ ${}_2C_2$	3 terms
Row 4	${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$	4 terms
Row 5	${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$	5 terms
Row 6	${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$	6 terms
Row 7	${}_6C_0$ ${}_6C_1$ ${}_6C_2$ ${}_6C_3$ ${}_6C_4$ ${}_6C_5$ ${}_6C_6$	7 terms
Row 8	${}_7C_0$ ${}_7C_1$ ${}_7C_2$ ${}_7C_3$ ${}_7C_4$ ${}_7C_5$ ${}_7C_6$ ${}_7C_7$	8 terms

3. The combination notation equals the term in the corresponding row of the triangle in question 1.
e.g., Row 5 ${}_4C_0 = 1$, ${}_4C_1 = 4$, ${}_4C_2 = 6$, ${}_4C_3 = 4$ and ${}_4C_4 = 1$

4. a. two terms b. five terms c. eight terms d. n terms

5. ${}_{13}C_0$ ${}_{13}C_1$ ${}_{13}C_2$ ${}_{13}C_3$ ${}_{13}C_4$ ${}_{13}C_5$ ${}_{13}C_6$ ${}_{13}C_7$ ${}_{13}C_8$ ${}_{13}C_9$ ${}_{13}C_{10}$ ${}_{13}C_{11}$ ${}_{13}C_{12}$ ${}_{13}C_{13}$

EXERCISE 6b

6.

Expanded Polynomial	Pascal's Triangle
$(x+y)^0 = 1$	Row 1: 1
$(x+y)^1 = x+y$	Row 2: 1 1
$(x+y)^2 = x^2+2xy+y^2$	Row 3: 1 2 1
$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$	Row 4: 1 3 3 1
$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^3$	Row 5: 1 4 6 4 1

7. a. i. 4 terms ii. 1 term

iii. 9 terms iv. 21 terms

v. $n + 1$ terms

b. The coefficients of the expansion of $(x + y)^2$ are the same as the **third** row of Pascal's Triangle.

c. The coefficients of the expansion of $(x + y)^4$ are the same as the **fifth** row of Pascal's Triangle.

d. The coefficients of the expansion of $(x + y)^n$ would be the same as the $n + 1$ row of Pascal's Triangle.

e. Each term is a fourth-degree term.

f. Each term would be an *n*th-degree term.

g. In the first term, x is raised to the **third** power and y is raised to the **0** power.
In the second term, x is raised to the **second** power and y is raised to the **first** power.
In the third term, x is raised to the **first** power and y is raised to the **second** power.
Last, in the fourth term, x is raised to the **0** power and y is raised to the **third** power.

h. In the first term, x is raised to the ***n*th** power and y is raised to the **0** power.
In the second term, x is raised to the $n - 1$ power and y is raised to the **first** power.
In the third term, x is raised to the $n - 2$ power and y is raised to the **second** power.
In the last term, x is raised to the **0** power and y is raised to the ***n*th** power.

i.
- There would be **8** terms.
- The coefficients of the terms would be the same as the **eighth** row of Pascal's Triangle.
- The first term is x^7 and the last term is y^7.

j. $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

8. a. 11 b. 4 c. 13 d. 5

9. a. 23 b. 101

10. a. $n = 3 + 4$ $= 7$ b. $n = 10 + 12$ $= 22$ c. $n = 7 + 1$ $= 8$ d. $n = 8 + 8$ $= 16$

11. a. $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

b. $(x+y)^9 = x^9 + 9x^8y + 36x^7y^2 + 84x^6y^3 + 126x^5y^4 + 126x^4y^5 + 84x^3y^6 + 36x^2y^7 + 9xy^8 + y^9$

EXERCISE 6c

12. a. $(2p+3q)^5 = (2p)^5 + 5(2p)^4(3q) + 10(2p)^3(3q)^2 + 10(2p)^2(3q)^3 + 5(2p)(3q)^4 + (3q)^5$
$= 32p^5 + 240p^4q + 720p^3q^2 + 1080p^2q^3 + 810pq^4 + 243q^5$

b. $(1-m)^6 = (1)^6 + 6(1)^5(-m) + 15(1)^4(-m)^2 + 20(1)^3(-m)^3 + 15(1)^2(-m)^4 + 6(1)(-m)^5 + (-m)^6$
$= 1 - 6m + 15m^2 - 20m^3 + 15m^4 - 6m^5 + m^6$

c. $(3a^2-2b)^4 = (3a^2)^4 + 4(3a^2)^3(-2b) + 6(3a^2)^2(-2b)^2 + 4(3a^2)(-2b)^3 + (-2b)^4$
$= 81a^8 - 216a^6b + 216a^4b^2 - 96a^2b^3 + 16b^4$

d. $\left(x^2 - \frac{2}{x}\right)^3 = (x^2)^3 + 3(x^2)^2\left(-\frac{2}{x}\right) + 3(x^2)\left(-\frac{2}{x}\right)^2 + \left(-\frac{2}{x}\right)^3$
$= x^6 - 6x^3 + 12 - \frac{8}{x^3}$

13. $(a+b)^6 - (a-b)^6$
$= \left[(a)^6 + 6(a)^5(b) + 15(a)^4(b)^2 + 20(a)^3(b)^3 + 15(a)^2(b)^4 + 6(a)(b)^5 + (b)^6\right]$
$\quad - \left[(a)^6 + 6(a)^5(-b) + 15(a)^4(-b)^2 + 20(a)^3(-b)^3 + 15(a)^2(-b)^4 + 6(a)(-b)^5 + (-b)^6\right]$
$= \left[a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6\right]$
$\quad - \left[a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6\right]$
$= 12a^5b + 40a^3b^3 + 12ab^5$

14. a. $(100+5)^2 = 100^2 + 2(100)(5) + 5^2$
$= 10\,000 + 1000 + 25$
$= 11\,025$

b. $(105)^3 = (100+5)^3 = 100^3 + 3(100)^2(5) + 3(100)(5)^2 + 5^3$
$= 1\,000\,000 + 150\,000 + 7500 + 125$
$= 1\,157\,625$

EXERCISE 6d

15. There are 84 different routes.

School 1 1 1 or $\frac{9!}{3!6!} = 84$ or ${}_9C_6 = 84$

1	2	3	4
1	3	6	10
1	4	10	20
1	5	15	35
1	6	21	56
1	7	28	84

House

16.

	Pascal's Triangle	Sum	2^n
Row 1	1	1	2^0
Row 2	1 + 1	2	2^1
Row 3	1 + 2 + 1	4	2^2
Row 4	1 + 3 + 3 + 1	8	2^3
Row 5	1 + 4 + 6 + 4 + 1	16	2^4
Row 6	1 + 5 + 10 + 10 + 5 + 1	32	2^5
Row 7	1 + 6 + 15 + 20 + 15 + 6 + 1	64	2^6
Row 8	1 + 7 + 21 + 35 + 35 + 21 + 7 + 1	128	2^7

17. $(x+y)^7 = {}_7C_0x^7 + {}_7C_1x^6y + {}_7C_2x^5y^2 + {}_7C_3x^4y^3 + {}_7C_4x^3y^4 + {}_7C_5x^2y^5 + {}_7C_6xy^6 + {}_7C_7y$

Let $x = y = 1$.

$\therefore 2^7 = {}_7C_0(1)^7 + {}_7C_1(1)^6(1) + {}_7C_2(1)^5(1)^2 + {}_7C_3(1)^4(1)^3 + {}_7C_4(1)^3(1)^4 + {}_7C_5(1)^2(1)^5$

$+ {}_7C_6(1)(1)^6 + {}_7C_7(1)^7$

$= 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1$

$= 128$

18. Number of different sums $= {}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5 = 5 + 10 + 10 + 5 + 1 = 31$, or one less than the sum of the sixth row of Pascal's Triangle.

19. There are four prime factors of 910: 2, 5, 7 and 13. Any two or more of these when multiplied should also be a factor. 1 and 910 are also positive factors of 910.

$\therefore$ number of different factors $= {}_4C_0 + {}_4C_1 + {}_4C_2 + {}_4C_3 + {}_4C_4$

$= 1 + 4 + 6 + 4 + 1$

$= 16$

20. a.

Row 1 $\quad {}_0C_0 = \underline{1}$

Row 2 $\quad {}_1C_0 = \underline{1} \quad {}_1C_1 = \underline{1}$

Row 3 $\quad {}_2C_0 = \underline{1} \quad {}_2C_1 = \underline{2} \quad {}_2C_2 = \underline{1}$

Row 4 $\quad {}_3C_0 = \underline{1} \quad {}_3C_1 = \underline{3} \quad {}_3C_2 = \underline{3} \quad {}_3C_3 = \underline{1}$

Row 5 $\quad {}_4C_0 = \underline{1} \quad {}_4C_1 = \underline{4} \quad {}_4C_2 = \underline{6} \quad {}_4C_3 = \underline{4} \quad {}_4C_4 = \underline{1}$

Row 6 $\quad {}_5C_0 = \underline{1} \quad {}_5C_1 = \underline{5} \quad {}_5C_2 = \underline{10} \quad {}_5C_3 = \underline{10} \quad {}_5C_4 = \underline{5} \quad {}_5C_5 = \underline{1}$

Row 7 $\quad {}_6C_0 = \underline{1} \quad {}_6C_1 = \underline{6} \quad {}_6C_2 = \underline{15} \quad {}_6C_3 = \underline{20} \quad {}_6C_4 = \underline{15} \quad {}_6C_5 = \underline{6} \quad {}_6C_6 = \underline{1}$

b. 4, 4 $\qquad$ c. ${}_6C_2, {}_5C_3$ $\qquad$ d. ${}_{10}C_7$, ${}_nC_{n-r}$

e.

L.S.	R.S.
$\dfrac{n!}{(n-r)!r!}$	${}_nC_{n-r}$ $\dfrac{n!}{(n-n+r)!(n-r)!}$ $= \dfrac{n!}{(n-r)!r!}$

L.S. = R.S.

f. ${}_3C_1 + {}_3C_2 = 6$

g. ${}_4C_2 = 6$

h. ${}_5C_2 + {}_5C_3 = {}_6C_3$

i. ${}_6C_3 + {}_6C_4 = {}_7C_4$

j. ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$

k.

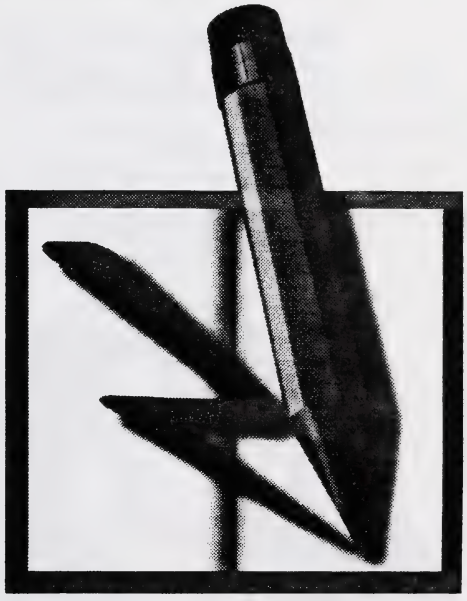
L.S.	R.S.
${}_{n+1}C_{r+1}$	${}_nC_r + {}_nC_{r+1}$
$= \dfrac{(n+1)!}{(n+1-r-1)!(r+1)!}$	$= \dfrac{n!}{(n-r)!r!} + \dfrac{n!}{(n-r-1)!(r+1)!}$
$= \dfrac{(n+1)!}{(n-r)!(r+1)!}$	$= \dfrac{n!(r+1)}{(n-r)!r!(r+1)} + \dfrac{n!(n-r)}{(n-r-1)!(r+1)!(n-r)}$
	$= \dfrac{n!(r+1)+n!(n-r)}{(n-r)!(r+1)!}$
	$= \dfrac{n!(r+1+n-r)}{(n-r)!(r+1)!}$
	$= \dfrac{(n+1)!}{(n-r)!(r+1)!}$

L.S. = R.S.

1997-401 cc v7

MATH FACTOR

PERMUTATIONS AND COMBINATIONS



Lesson 7: Binomial Theorem

Student Workbook

Produced by

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Alberta EDUCATION

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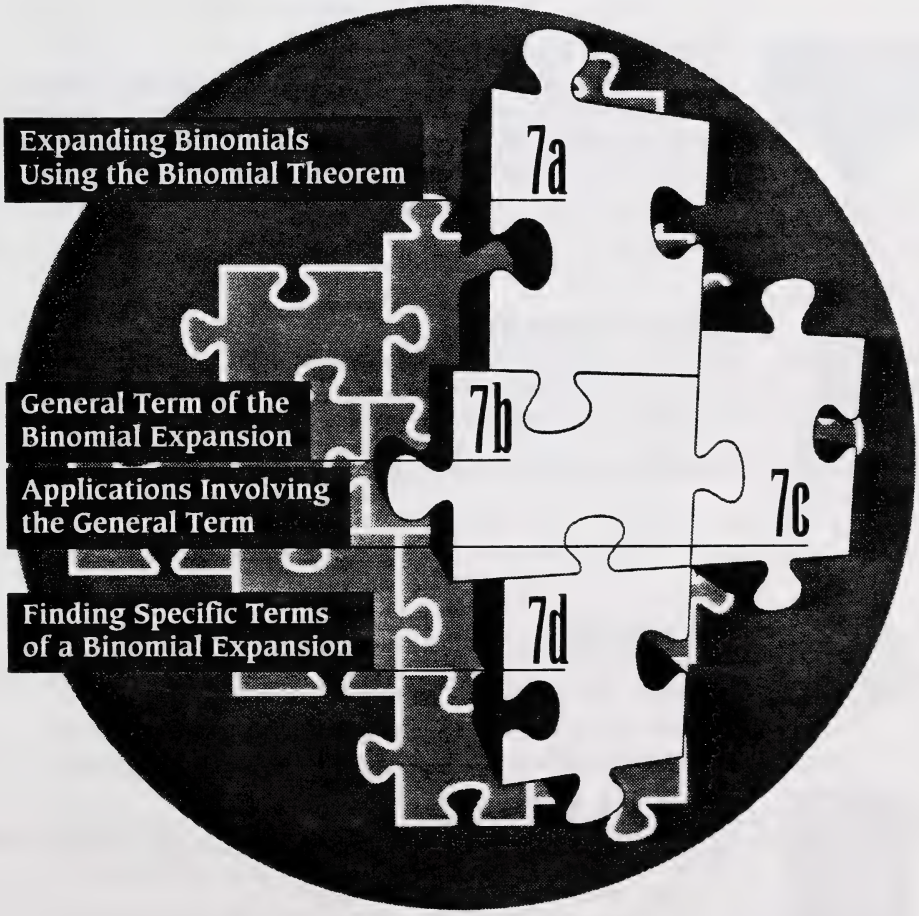
Lesson 7: BINOMIAL THEOREM

LEARNER PECTATIONS

By the end of on 7 you should be able to

- expand binomials of the form $(x + y)^n$, where n is a whole number, using the Binomial Theorem
- relate the numerical coefficients in a binomial expansion to the terms of Pascal's Triangle and vice versa

is lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Expanding Binomials Using the Binomial Theorem

Challenge yourself

Expand the expression $(a+1)^5$.

MATH FACTOR

Watch the Introduction and Video Segment 7a.

Recall the following:

- Pascal's Triangle can be written two ways.

Row 1	1	${}_0C_0$
Row 2	1 1	${}_1C_0$ ${}_1C_1$
Row 3	1 2 1	${}_2C_0$ ${}_2C_1$ ${}_2C_2$
Row 4	1 3 3 1	${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$
Row 5	1 4 6 4 1	${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$

- The coefficient of the terms of the expansion of the binomial $(x+y)^n$ corresponds to Pascal's Triangle.

$(x+y)^0 =$	1	${}_0C_0$
$(x+y)^1 =$	$1x + 1y$	${}_1C_0x + {}_1C_1y$
$(x+y)^2 =$	$1x^2 + 2xy + 1y^2$	${}_2C_0x^2 + {}_2C_1xy + {}_2C_2y^2$
$(x+y)^3 =$	$1x^3 + 3x^2y + 3xy^2 + y^3$	${}_3C_0x^3 + {}_3C_1x^2y + {}_3C_2xy^2 + {}_3C_3y^3$
$(x+y)^4 =$	$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$	${}_4C_0x^4 + {}_4C_1x^3y + {}_4C_2x^2y^2 + {}_4C_3xy^3 + {}_4C_4y^4$

Note the patterns in the expansion of $(x+y)^n$:

- The number of terms in the expansion is $n + 1$. For example, when $(x+y)^2$ is expanded, there are three terms.
- The degree (exponent or sum of exponents) of each term of the expansion is n. For example, when $(x+y)^3$ is expanded, each term is a third-degree term.
- In each expansion, the first term x is raised to the nth power and y is raised to the zero power. In each successive term, the exponent of x is decreased by one and the exponent of y is increased by one.
- The coefficients of the expansion correspond to the $n + 1$ row of Pascal's Triangle .

- The coefficients, in combination notation, are also significant.

  e.g., $(x+y)^4 = {}_4C_0x^4 + {}_4C_1x^3y + {}_4C_2x^2y^2 + {}_4C_3xy^3 + {}_4C_4y^4$

  In ${}_4C_0x^4$, the 4 in ${}_4C_0$ is the power to which the binomial is raised. The 0 represents the exponent of y in the term. The exponent of x is $4-0$.

In general, the **Binomial Theorem** is as follows:

$$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_kx^{n-k}y^k + \ldots + {}_nC_{n-2}x^2y^{n-2} + {}_nC_{n-1}xy^{n-1} + {}_nC_ny^n, \text{ where } n \in W$$

The general term of $(x+y)^n$ can be expressed as ${}_nC_kx^{n-k}y^k$.

Example 1

Expand $(x+y)^6$ using the Binomial Theorem.

Solution

$$\begin{aligned}(x+y)^6 &= {}_6C_0x^6 + {}_6C_1x^5y + {}_6C_2x^4y^2 + {}_6C_3x^3y^3 + {}_6C_4x^2y^4 + {}_6C_5xy^5 + {}_6C_6y^6\\ &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6\end{aligned}$$

Example 2

Expand $(2m-3)^3$ using the Binomial Theorem.

Solution

$(2m-3)^3$ will expand the same as $(x+y)^3$.

$$(x+y)^3 = {}_3C_0x^3 + {}_3C_1x^2y + {}_3C_2xy^2 + {}_3C_3y^3$$

$$\begin{aligned}\therefore (2m-3)^3 &= {}_3C_0(2m)^3 + {}_3C_1(2m)^2(-3) + {}_3C_2(2m)(-3)^2 + {}_3C_3(-3)^3\\ &= 8m^3 + 3(4m^2)(-3) + 3(2m)(9) - 27\\ &= 8m^3 - 36m^2 + 54m - 27\end{aligned}$$

Example 3

If the first three terms of $(x+y)^{11}$ are ${}_{11}C_0 x^{11} + {}_{11}C_1 x^{10} y + {}_{11}C_2 x^9 y^2$, then write the first three terms of $\left(a^2 - \frac{5}{a}\right)^{11}$.

Solution

$$\therefore \left(a^2 - \frac{5}{a}\right)^{11} = {}_{11}C_0 \left(a^2\right)^{11} + {}_{11}C_1 \left(a^2\right)^{10} \left(-\frac{5}{a}\right) + {}_{11}C_2 \left(a^2\right)^9 \left(-\frac{5}{a}\right)^2$$

$$= a^{22} + 11\left(a^{20}\right)\left(-\frac{5}{a}\right) + 55\left(a^{18}\right)\left(\frac{25}{a^2}\right)$$

$$= a^{22} - 55a^{19} + 1375a^{16}$$

The first three terms of $\left(a^2 - \frac{5}{a}\right)^{11}$ are $a^{22} - 55a^{19} + 1375a^{16}$.

TEXT

Additional Examples
Mathematics 12: p. 510
Mathematics: Principles and Process 12: p. 424
Holtmath 12: p. 362 (Example 2)

Exercises

1. Use the Binomial Theorem to expand the following:

   a. $(x+1)^7$

   b. $(x-y)^6$

   c. $\left(2a^3 + 3\right)^5$

   d. $\left(x^2 - \frac{1}{x}\right)^4$

2. If the first four terms of $(x+y)^{11}$ are ${}_{11}C_0 x^{11} + {}_{11}C_1 x^{10} y + {}_{11}C_2 x^9 y^2 + {}_{11}C_3 x^8 y^3$, then write the first four terms of the following:

   a. $(2x-3)^{10}$

b. $\left(1-3m^2\right)^9$

c. $\left(3a^3-\frac{1}{a^2}\right)^{20}$

3. If the last three terms of $(x+y)^{11}$ are ${}_{11}C_9x^2y^9 + {}_{11}C_{10}xy^{10} + {}_{11}C_{11}y^{11}$, then write the last three terms of the following:

a. $(x-2y)^{13}$

b. $\left(x+\frac{1}{y}\right)^{19}$

4. Use the Binomial Theorem to expand $(a+b-c)^3$.

EXERCISE 7a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 512 (Questions 1, 4-10)
Mathematics: Principles and Process 12: p. 425 (Questions 1-3, 5)
Holtmath 12: p. 362 (Questions 9, 11)

General Term of the Binomial Expansion

Challenge yourself

Find the fourth-last term in the expansion of $(c+d)^{10}$.

MATH FACTOR

Watch Video Segment 7b.

The general term of the Binomial Theorem allows a specific term to be found without finding the preceding terms.

If $(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_kx^{n-k}y^k + \ldots$
$+ {}_nC_{n-2}x^2y^{n-2} + {}_nC_{n-1}xy_{n-1} + {}_nC_ny^n$, where $n \in W$, then the general term is $t_{k+1} = {}_nC_kx^{n-k}y^k$, where $k \leq n$.

Example 1

Find the fourth term, t_4, of $\left(3p - \frac{1}{2q^2}\right)^7$.

Solution

Step 1: Determine the value of k, n, x and y.

$t_{k+1} = t_4$, $n = 7$, $x = 3p$ and $y = -\frac{1}{2q^2}$

$\therefore k = 3$

Step 2: Substitute into the general term.

$$t_{k+1} = {}_nC_kx^{n-k}y^k$$

$$t_4 = {}_7C_3(3_p)^4\left(-\frac{1}{2q^2}\right)^3$$

$$= 35\left(81p^4\right)\left(-\frac{1}{8q^6}\right)$$

$$= -\frac{2835p^4}{86q^6}$$

Thus, the fourth term is $-\frac{2835p}{8q^6}$.

Example 2

Find the sixth term of $\left(3x^2 - \frac{1}{x^4}\right)^{10}$.

Solution

$t_{k+1} = t_6$, $n = 10$, $x = 3x^2$ and $y = -\frac{1}{x^4}$

$\therefore k = 5$

$$t_{k+1} = {}_nC_k x^{n-k} y^k$$

$$t_6 = {}_{10}C_5 \left(3x^2\right)^5 \left(-\frac{1}{x^4}\right)^5$$

$$= 252\left(243x^{10}\right)\left(-\frac{1}{x^{20}}\right)$$

$$= -\frac{61\,236}{x^{10}}$$

Thus, the sixth term is $-\frac{61\,236}{x^{10}}$.

Example 3

In the expansion of $(x + y)^{12}$, determine which term contains $x^5 y^7$ and then determine the numerical coefficient of that term in the form ${}_nC_k$.

Solution

$x^5 y^7$ is the eighth term in the expansion $(7 + 1 = 8)$.
The coefficient of the eighth term is ${}_{12}C_7$.

TEXT

Additional Example
Mathematics 12: p. 511 (Example 1)

Exercises

5. Find the indicated term.

   a. the sixth term of $(a+b)^{12}$

   b. t_8 of $(3-b)^{11}$

   c. t_{14} of $\left(4p-q^2\right)^{15}$

   d. t_4 of $\left(3x^2-\frac{y^3}{6}\right)^6$

6. In the expansion of $(x+y)^{16}$, answer the following:

   a. Which term contains xy^{15}?

b. What is its coefficient in the form ${}_nC_k$?

7. ${}_{12}C_{r-1}x^3y^{r-1}$ is a term in the expansion of $(x+y)^{12}$. What is the value of r?

8. The formula for the general term is $t_{k+1} = {}_nC_k x^{n-k} y^k$. Rewrite this formula so that t_k is the left side.

EXERCISE 7b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 512 (Questions 2, 11)
Mathematics: Principles and Process 12: p. 425 (Question 6)
Holtmath 12: p. 362 (Questions 12a, 12b, 12c)

Applications Involving the General Term

A multiple-choice test has ten questions, each of which has four responses. What is the probability of guessing half the questions correctly on this test?

Watch Video Segment 7c.

The Binomial Theorem can be used to solve the following types of problems:

- problems in which there are only two possible outcomes
- problems in which events are independent (i.e., the probability of each outcome is the same for each trial). Examples include guessing right or guessing wrong, tossing a head or a tail, having a boy or a girl, a switch that can be turned on or off, or bits of information on a computer.

Such problems are often called Bernouilli problems or trials. The probability of a Bernouilli trial can be found using the general term of a binomial term ${}_nC_k x^{n-k} y^k$, where k represents the number of successes and n represents the number of trials.

Example 1

A multiple-choice test has ten questions, each of which has four responses. What is the probability of guessing exactly four correct answers?

Solution

The probability of guessing correctly is $\frac{1}{4}$ and of guessing incorrectly is $\frac{3}{4}$. The number of correct answers is four; so, $k = 4$ and $n = 10$.

$$\begin{aligned} P(\text{4 correct answers}) &= {}_{10}C_4 \left(\frac{3}{4}\right)^6 \left(\frac{1}{4}\right)^4 \\ &= 210(0.75)^6 (0.25)^4 \\ &= 0.145 \end{aligned}$$

Therefore, the probability of guessing four correct answers is 0.145.

Example 2

A fair coin is tossed five times. What is the probability of getting four heads and one tail?

Solution

$P(\text{head}) = \frac{1}{2}$ and $P(\text{tail}) = \frac{1}{2}$. Also, $n = 5$ as there are five tosses, and $k = 1$ as one successful trial is a tail.

$$P(4H, 1T) = {}_5C_1\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1$$
$$= 5\left(\frac{1}{2}\right)^5$$
$$= \frac{5}{32}$$

Therefore, the probability of four heads and one tail is $\frac{5}{32}$.

Note: This problem could have been solved using $k = 4$ to represent that four successful trials are heads.

TEXT

Additional Example
Mathematics: Principles and Process 12: pp. 426, 427

Exercises

9. A true-and-false test has 20 questions. What is the probability of getting 60% by guessing the answers?

EXCELLENCE

10. The midterm exam has 30 multiple-choice questions. Each question has four responses. If a student randomly guesses at the answers, determine the probability of the following:

    a. the student will pass the exam

b. the student will answer exactly 12 questions correctly

11. If there is an equal chance of a boy or girl being born to a family, then determine the probability of a family with six children having four boys and two girls.

12. A fair coin is tossed 15 times. Find the probability of the following:

    a. 12 heads occurring

    b. 8 tails occurring

EXERCISE 7c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 513 (Question 15)
Mathematics: Principles and Process 12: p. 428 (Questions 2, 3, 4a, 4b, 4c, 4d, 5)
Holtmath 12: p. 367 (Question 9)

Finding Specific Terms of a Binomial Expansion

Challenge yourself

Without expanding the binomial $\left(a^2+\frac{1}{a}\right)^9$, determine the term that has a^6.

MATH FACTOR

Watch Video Segment 7d.

When finding a specific term without expanding a binomial, use the general term $t_{k+1} = {}_nC_k x^{n-k} y^k$.

Example 1

Which term is independent of x in the expansion of the binomial $\left(x^2-\frac{2}{x}\right)^6$?

Solution

In the general term, substitute $n=6$, $x=x^2$, $y=-\frac{2}{x}$ and k is unknown.

$$\begin{aligned} t_{k+1} &= {}_nC_k x^{n-k} y^k \\ &= {}_6C_k \left(x^2\right)^{6-k} \left(\frac{-2}{x}\right)^k \\ &= {}_6C_k \left(x^{12-2k}\right)\left(-2^k x^{-k}\right) \\ &= {}_6C_k (-2)^k x^{12-3k} \end{aligned}$$

In order for the term to be independent of x, x^{12-3k} must equal x^0.
If $x^{12-3k} = x^0$, then $12-3k=0$ or $k=4$.
If $k=4$, then the t_{4+1} or t_5 (the fifth term) is independent of x.

Example 2

Which term of the binomial $\left(3p^2 - 2\right)^{10}$ involves p^2, and what is the term?

Solution

In the general term, substitute $n = 10$, $x = 3p^2$, $y = -2$ and k is unknown.

$$\begin{aligned} t_{k+1} &= {}_nC_k x^{n-k} y^k \\ &= {}_{10}C_k \left(3p^2\right)^{10-k} (-2)^k \\ &= {}_{10}C_k \left(3^{10-k}\right)\left(p^{20-2k}\right)(-2)^k \end{aligned}$$

In order for the term to involve p^2, p^{20-2k} must equal p^2.

If $p^{20-2k} = p^2$, then $20 - 2k = 2$ or $k = 9$.

If $k = 9$, then the t_{9+1}, or t_{10} (the tenth term) involves p^2.

$$\begin{aligned} t_{10} &= {}_{10}C_9 \left(3p^2\right)(-2)^9 \\ &= 10\left(3p^2\right)(-512) \\ &= -15\,360p^2 \end{aligned}$$

Therefore, the tenth term is $-15\,360p^2$.

TEXT

Additional Example
Mathematics 12: p. 511 (Example 2)

Exercises

13. If the numerical coefficient is not stated, $a^x b^7$ is a term in the expansion of $(a+b)^{12}$.

    a. What is the value of x?

b. Which term involves $a^x b^7$?

c. What is the coefficient of this term?

14. What is the numerical coefficient for the term involving $a^9 b^7$ in the expansion of $(a-b)^{16}$?

15. Determine the fifth term of $\left(x^2 - \sqrt{y}\right)^8$.

16. Under what condition does $(x+y)^n$ have a middle term?

17. Which term in the expansion of $\left(\frac{1}{2}x - y\right)^6$ is $\frac{15}{16}x^4y^2$?

18. Without expanding the binomial, find the following:

a. the term involving x^3 in $\left(x^3 + \frac{2}{x}\right)^5$

b. the term independent of a in $\left(3a^2 - \frac{1}{a}\right)^9$

EXERCISE 7d SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 512 (Question 12)
Holtmath 12: p. 362 (Question 12e)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Expanding binomials using the Binomial Theorem and finding a specific term of a binomial expansion are important concepts to be mastered. A complete review of permutations, combinations and the Binomial Theorem is included in **Lesson 8: Review**.

Solutions

EXERCISE 7a

1. a. $(x+1)^7 = {}_7C_0x^7 + {}_7C_1x^6(1) + {}_7C_2x^5(1)^2 + {}_7C_3x^4(1)^3 + {}_7C_4x^3(1)^4 + {}_7C_5x^2(1)^5$
$+ {}_7C_6x(1)^6 + {}_7C_7(1)^7$
$= x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

b. $(x-y)^6 = {}_6C_0x^6 + {}_6C_1x^5(-y) + {}_6C_2x^4(-y)^2 + {}_6C_3x^3(-y)^3 + {}_6C_4x^2(-y)^4$
$+ {}_6C_5x(-y)^5 + {}_6C_6(-y)^6$
$= x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$

c. $\left(2a^3+3\right)^5 = {}_5C_0\left(2a^3\right)^5 + {}_5C_1\left(2a^3\right)^4(3) + {}_5C_2\left(2a^3\right)^3(3)^2 + {}_5C_3\left(2a^3\right)^2(3)^3$
$+ {}_5C_4\left(2a^3\right)(3)^4 + {}_5C_5(3)^5$
$= 32a^{15} + 240a^{12} + 720a^9 + 1080a^6 + 810a^3 + 243$

d. $\left(x^2 - \frac{1}{x}\right)^4 = {}_4C_0\left(x^2\right)^4 + {}_4C_1\left(x^2\right)^3\left(-\frac{1}{x}\right) + {}_4C_2\left(x^2\right)^2\left(-\frac{1}{x}\right)^2 + {}_4C_3\left(x^2\right)\left(-\frac{1}{x}\right)^3$
$+ {}_4C_4\left(-\frac{1}{x}\right)^4$
$= x^8 - 4x^5 + 6x^2 - \frac{4}{x} + \frac{1}{x^4}$

2. a. $(2x-3)^{10} = {}_{10}C_0(2x)^{10} + {}_{10}C_1(2x)^9(-3) + {}_{10}C_2(2x)^8(-3)^2 + {}_{10}C_3(2x)^7(-3)^3 + \ldots$
$= 11\,024x^{10} - 15\,360x^9 + 103\,680x^8 - 414\,720x^7 \ldots$

b. $\left(1-3m^2\right)^9 = {}_9C_0(1)^9 + {}_9C_1(1)^8\left(-3m^2\right) + {}_9C_2(1)^7\left(-3m^2\right)^2 + {}_9C_3(1)^6\left(-3m^2\right)^3 + \ldots$
$= 1 - 27m^2 + 324m^4 - 2268m^6 \ldots$

c. $\left(3a^3 - \frac{1}{a^2}\right)^{20} = {}_{20}C_0\left(3a^3\right)^{20} + {}_{20}C_1\left(3a^3\right)^{19}\left(-\frac{1}{a^2}\right) + {}_{20}C_2\left(3a^3\right)^{18}\left(-\frac{1}{a^2}\right)^2$
$+ {}_{20}C_3\left(3a^3\right)^{17}\left(-\frac{1}{a^2}\right)^3 + \ldots$
$= 3^{20}a^{60} - 20\left(3^{19}\right)a^{55} + 190\left(3^{18}\right)a^{50} - 1140\left(3^{17}\right)a^{45} + \ldots$

3. a. $(x-2y)^{13} = \ldots + {}_{13}C_{11}(x)^2(-2y)^{11} + {}_{13}C_{12}(x)(-2y)^{12} + {}_{13}C_{13}(-2y)^{13}$
$= \ldots - 78(2)^{11}x^2y^{11} + 13(2)^{12}xy^{12} - (2)^{13}y^{13}$

b. $\left(x^2+\frac{1}{y}\right)^{19} = \ldots + {}_{19}C_{17}(x)^2\left(\frac{1}{y}\right)^{17} + {}_{19}C_{18}(x)\left(\frac{1}{y}\right)^{18} + {}_{19}C_{19}\left(\frac{1}{y}\right)^{19}$

$= \ldots + 171\frac{x^2}{y^{17}} + 19\frac{x}{y^{18}} + \frac{1}{y^{19}}$

4. $(a+b-c)^3 = \left[(a+b)-c\right]^3$

$= {}_3C_0(a+b)^3 + {}_3C_1(a+b)^2(-c) + {}_3C_2(a+b)(-c)^2 + {}_3C_3(-c)^3$

$= (a+b)^3 - 3c(a+b)^2 + 3c^2(a+b) - c^3$

$= a^3 + 3a^2b + 3ab^2 + b^3 - 3c\left(a^2 + 2ab + b^2\right) + 3ac^2 + 3bc^2 - c^3$

$= a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3$

or

$= a^3 + b^3 - c^3 + 3a^2b - 3a^2c + 3ab^2 + 3ac^2 - 6abc - 3b^2c + 3bc^2$

EXERCISE 7b

5. a. $t_{k+1} = t_6,\ n = 12,\ x = a \text{ and } y = b$

$k = 5$

$t_{k+1} = {}_nC_k x^{n-k} y^k$

$t_6 = {}_{12}C_5(a)^7(b)^5$

$t_6 = 792a^7b^5$

b. $t_{k+1} = t_8,\ n = 11,\ x = 3 \text{ and } y = -b$

$k = 7$

$t_{k+1} = {}_nC_k x^{n-k} y^k$

$t_8 = {}_{11}C_7(3)^4(-b)^7$

$t^8 = -26\,730b^7$

c. $t_{k+1} = t_{14},\ n = 15,\ x = 4p \text{ and } y = -q^2$

$k = 13$

$t_{k+1} = {}_nC_k x^{n-k} y^k$

$t_{14} = {}_{15}C_{13}(4p)^2\left(-q^2\right)^{13}$

$t_{14} = -1680p^2q^{26}$

d. $t_{k+1} = {}_nC_k x^{n-k} y^k$

$$t_4 = {}_6C_3 \left(3x^2\right)^3 \left(-\frac{y^3}{6}\right)^3$$

$$t_4 = -\frac{5}{2} x^6 y^9$$

6. a. 16th b. ${}_{16}C_{15}$

7. In the term ${}_{12}C_{r-1} x^3 y^{r-1}$, $3 + r - 1 = 12$; so, $r = 10$.

8. $t_{k+1-1} = {}_nC_{k-1} x^{n-(k-1)} y^{k-1}$

$t_k = {}_nC_{k-1} x^{n-k+1} y^{k-1}$

EXERCISE 7c

9. In order to get 60% on the test, a person must guess 12 questions correctly.

$\mathrm{P(correct)} = \frac{1}{2}$, $\mathrm{P(incorrect)} = \frac{1}{2}$, $n = 20$ and $k = 12$

$$\mathrm{P(60\% \text{ on test})} = {}_{20}C_{12} \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{12}$$

$$= \frac{125\,970}{2^{20}}$$

$\doteq 0.12$ or 12% chance

10. a. In order to pass (50%), the student must guess 15 questions correctly.

$\mathrm{P(correct)} = \frac{1}{4}$, $\mathrm{P(incorrect)} = \frac{3}{4}$, $n = 30$ and $k = 15$

$$\mathrm{P(passing\ exam)} = {}_{30}C_{15} \left(\frac{3}{4}\right)^{15} \left(\frac{1}{4}\right)^{15}$$

$\doteq 0.0019$ or 0.19% chance

b. $\mathrm{P(correct)} = \frac{1}{4}$, $\mathrm{P(incorrect)} = \frac{3}{4}$, $n = 30$ and $k = 12$

$$\mathrm{P(exactly\ 12\ correct)} = {}_{30}C_{12} \left(\frac{3}{4}\right)^{18} \left(\frac{1}{4}\right)^{12}$$

$\doteq 0.029$ or a 2.9% chance

11. Assume $P(\text{girl}) = P(\text{boy}) = \frac{1}{2}$; then $n = 6$ and $k = 4$ (4 successes in 6 trials).

$$P(\text{4 boys and 2 girls}) = {}_6C_4\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4$$
$$= \frac{15}{64}$$
$$\doteq 0.234$$

12. a. $P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$, $n = 15$ and $k = 12$ (12 successes in 15 trials)

$$P(\text{12 heads in 15 tosses}) = {}_{15}C_{12}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{12}$$
$$= \frac{455}{2^{15}}$$
$$\doteq 0.0139$$

b. $P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$, $n = 15$ and $k = 8$ (8 successes in 15 trials)

$$P(\text{8 tails in 15 tosses}) = {}_{15}C_8\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^8$$
$$= \frac{6435}{2^{15}}$$
$$\doteq 0.1964$$

EXERCISE 7d

13. a. $x = 5$

b. $k = 7$, so $a^x b^7$ would be the $7 + 1 =$ eighth term.

c. ${}_{12}C_7 = 792$

14. The term is ${}_{16}C_7 (a)^9 (-b)^7$, so the coefficient is $-11\,440$.

15. $t_{k+1} = t_5$, $n = 8$, $x = x^2$ and $y = -\sqrt{y}$

$k = 4$

$t_{k+1} = {}_nC_k x^{n-k} y^k$

$t_5 = {}_8C_4 \left(x^2\right)^4 \left(-\sqrt{y}\right)^4$

$t_5 = 70x^8 y^2$

16. $(x + y)^n$ has a middle term when n is an even positive integer.

17. $k = 2$ in the term $\frac{15}{16}x^4 y^2$, so this would be the $2 + 1$ or third term of the expansion of $\left(\frac{1}{2}x - y\right)^6$.

18. a. The general term of the expansion of $\left(x^3+\frac{2}{x}\right)^5$ is ${}_5C_k\left(x^3\right)^{5-k}\left(\frac{2}{x}\right)^k$ or ${}_5C_k(2)^k\left(x^3\right)^{5-k}\left(\frac{1}{x}\right)^k$.

$$\therefore \left(x^3\right)^{5-k}\left(\frac{1}{x}\right)^k = x^3$$
$$\left(x^{15-3k}\right)\left(x^{-k}\right) = x^3$$
$$x^{15-4k} = x^3$$

$\therefore 15-4k=3$ or $k=3$. If $k=3$, then the term involving x^3 is the fourth term $(3+1)$.

$$\therefore t_4 = {}_5C_3(2)^3\left(x^3\right)^2\left(\frac{1}{x}\right)^3 = 80x^3$$

b. The general term of the expansion of $\left(3a^2-\frac{1}{a}\right)^9$ is ${}_9C_k\left(3a^2\right)^{9-k}\left(-\frac{1}{a}\right)^k$ or ${}_9C_k(3)^{9-k}(-1)^k\left(a^2\right)^{9-k}\left(\frac{1}{a}\right)^k$.

In order for the term to independent of a, then $\left(a^2\right)^{9-k}\left(\frac{1}{a}\right)^k = a^0$

$$\left(a^{18-2k}\right)\left(a^{-k}\right) = a^0$$
$$a^{18-3k} = a^0$$
$$\therefore 18-3k=0 \text{ or } k=6$$

If $k=6$, then the seventh term is independent of a and $t_7 = {}_9C_6(3)^3(-1)^6\left(a^2\right)^3\left(\frac{1}{a}\right)^6 = 2268$.

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MATH FACTOR

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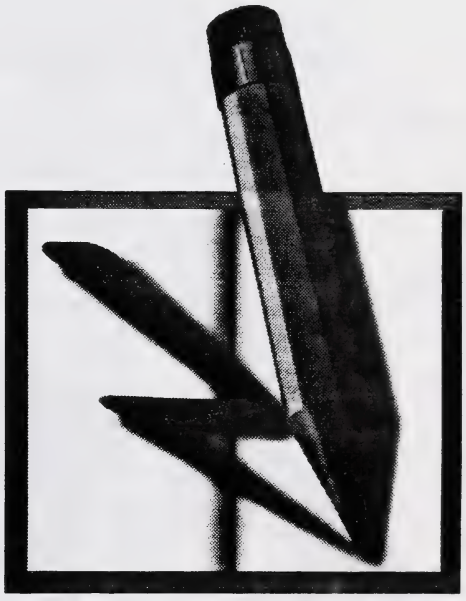
PERMUTATIONS AND COMBINATIONS



Lesson 8: Review

Student Workbook

Produced by

THE EDUCATION STATION

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for

Alberta EDUCATION

This guide accompanies **Review** from the Math Factor series.

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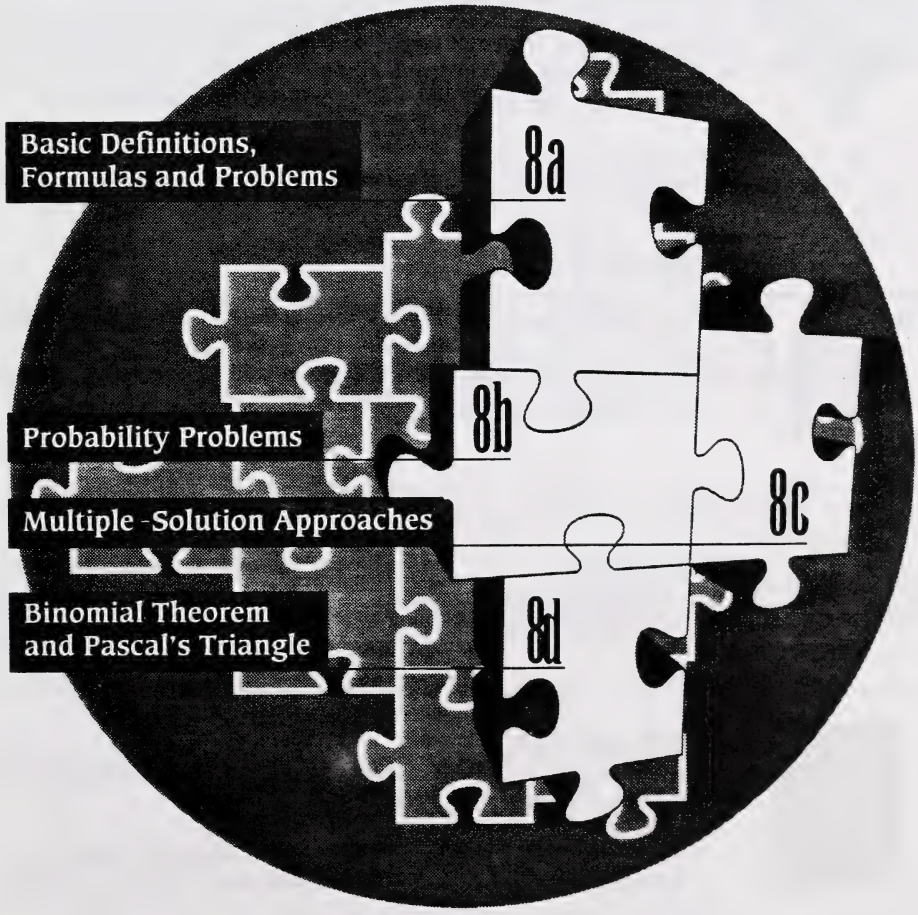
Lesson 8: REVIEW

LEARNER EXPECTATIONS

By the end of Lesson 8 you should be able to

- compare and contrast permutations and combinations
- demonstrate an understanding of the Fundamental Counting Principle
- solve problems involving linear permutations, permutations with restrictions and repetitions, circular permutations and permutations with reflections
- solve problems involving combinations, including probability problems that involve the use of combinations
- demonstrate how the numerical coefficients of the terms of a binomial expansion can be found by using the Binomial Theorem, including the general term

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Basic Definitions, Formulas and Problems

In Video Segment 8a, the students and the teacher discuss the basic concepts and formulas. Included in the discussion are the following:

- the Fundamental Counting Principle (FCP)
- linear, circular and ring permutations
- permutations with repetitions and/or restrictions
- combinations

Watch the Introduction and Video Segment 8a.

The **Fundamental Counting Principle (FCP)** is the basic multiplication rule upon which much permutation, combination and probability work is based.

Fundamental Counting Principle

If a first task can be done in k_1 ways, a second task in k_2 ways, a third task in k_3 ways, and so on, then the total number of possible ways the multiple tasks can be performed is $k_1 \times k_2 \times k_3 \times \ldots$.

Example 1

The first digit in a telephone number cannot be a 0 or 1. How many seven-digit telephone numbers are possible?

Solution

There are eight possibilities for the first digit {2, 3, 4, 5, 6, 7, 8, 9}. There are ten possibilities for the second digit {0, 1, 2, 3,..., 9}. There are also ten possibilities for the other digits.

Therefore, there are $8 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 8\,000\,000$ telephone numbers.

Linear, circular and ring permutations are arrangements in which the order of the objects must be considered. Tree diagrams and FCP can often be used to solve simple permutation problems that involve a small number of objects. In other cases, a formula approach is recommended.

Permutations: linear permutations without restrictions or repetitions

$${}_nP_r = \frac{n!}{(n-r)!}$$

Circular Permutations: the number of permutations of n objects in a circle $= (n-1)!$

Ring Permutations: the number of permutations of n objects on a ring $= \frac{(n-1)!}{2}$

Example 2

If no digit may be repeated in a number, how many different three-digit numbers can be made from the digits 1, 2, 3, 4, 5 and 6?

Solution

$${}_nP_r = {}_6P_3$$
$$= 120$$

Example 3

In how many different orders can six different keys be arranged on a key ring?

Solution

$$n = 6 \quad \therefore \frac{(n-1)!}{2} = \frac{(6-1)!}{2}$$
$$= \frac{5!}{2}$$
$$= 60 \text{ different orders}$$

Permutations with repetitions and/or restrictions often occur in problem-solving situations. The usual method for dealing with these situations is to deal with the restrictions **first** and apply the permutations rule for repetitions as required.

Permutation with Repetitions

The number of arrangements of n objects, where a of one kind are alike, b of another kind are alike, c of another kind are alike and so on, is $\frac{n!}{a!b!c!}$.

Example 4

Find the number of permutations of the letters in the word **PRINTER** for the following:

- there are no restrictions
- the first two letters must be vowels

Solution

- there are no restrictions

$n = 7$ and $a = 2$

$$\therefore \frac{n!}{a!} = \frac{7!}{2!}$$

$= 720$ permutations

- the first two letters must be vowels

The two vowels, I and E, must come first. They can be arranged in two different ways: IE and EI. The remaining five consonants can be arranged in $\frac{5!}{2!}$ ways. (The R occurs twice.)

So, there are $2 \times \frac{5!}{2!} = 5!$

$= 120$ permutations

Combinations are selections of objects where order or arrangement is not important. (Once a set of objects has been chosen, changing the arrangement of the objects **does not** result in a new combination.)

Combinations

The number of combinations of n objects taken r objects at a time is as follows:

$$_nC_r = \frac{n!}{(n-r)!r!}$$

Note: The formula does not apply unless the objects are distinct and there are no repetitions.

Example 5

In how many ways can a jury of 12 members and a set of 6 alternate jurors be chosen from a pool of 20 individuals? Leave the answer in factorial form.

Solution

The 12 jurors can be chosen in ${}_{20}C_{12}$ ways. The 6 alternate jurors can be chosen in ${}_{8}C_{6}$ ways.

Altogether, the number of possible selections $= {}_{20}C_{12} \times {}_{8}C_{6}$

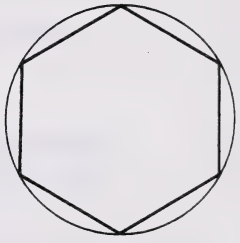
$$= \frac{20!}{(20-12)!12!} \times \frac{8!}{(8-6)!2!}$$

$$= \frac{20!}{8!12!} \times \frac{8!}{2!2!}$$

$$= \frac{20!}{12!2!2!}$$

Example 6

A hexagon is drawn inside a circle.
How many diagonals can be drawn to join the vertices of the hexagon?



Solution

A hexagon has 6 sides and 6 vertices.
The 6 vertices can be joined in ${}_{6}C_{2}$ ways.

$${}_{6}C_{2} = \frac{6!}{(6-2)!2!}$$

$$= \frac{6!}{4!2!}$$

$$= 15$$

But 6 of the 15 are existing sides of the hexagon. So, there are $15 - 6 = 9$ diagonals.

TEXT

Additional Examples
Mathematics 12: pp. 484-500 (Study all the examples.)
Mathematics: Principles and Process 12: pp. 402-415 (Study all the examples.)
Holtmath 12: pp. 350-360 (Study all the examples.)

Exercises

1. Determine the number of arrangements of all the letters of the word **SASSY** if each arrangement must begin with the following:

   a. the letter S

   b. exactly one S

2. a. A committee room containing only six chairs is to be used by eight members. In how many ways can the chairs be assigned to the members?

   b. In how many ways can the members now be seated if two more chairs are brought in?

3. If a hand consists of 5 cards, then how many different hands of the same suit can be dealt from a standard deck of 52 cards?

EXERCISE 8a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 517 (Questions 2, 4, 6, 8, 10)
Mathematics: Principles and Process 12: p. 431 (Questions 3-5)
Holtmath 12: p. 373 (Questions 6-11)

Probability Problems

An entire committee of ten is randomly arranged in a line for a group photo. What is the probability that the president and vice president stand together?

Probability can be defined in terms of favourable outcomes, favourable outcomes being those cases for which the probability is being calculated.

$$\text{probability} = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

The Fundamental Counting Principle can be written in modified form to apply to probability.

If the probability of an event occurring is p, the probability of a second event occurring is q and the probability of a third event occurring is r and so on, then the probability of the events all occurring is $p \times q \times r \times \ldots$

Example 1

A four-digit code is made by randomly selecting from the digits 1, 2, 3, 4 and 5. If no repetitions can be in a code, then what is the probability that the code is 5432?

Solution

There are ${}_5P_4$ possible arrangements of the digits.

$$\begin{aligned} {}_5P_4 &= \frac{5!}{(5-4)!} \\ &= \frac{5!}{1!} \\ &= 120 \end{aligned}$$

Only one of these arrangements, 5432, is favourable.

$$\begin{aligned} \text{probability} &= \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} \\ &= \frac{1}{{}_5P_4} \\ &= \frac{1}{120} \end{aligned}$$

Example 2

If 2 cards are picked, one after another, from a standard deck of 52 cards, use combinations to find the probability that they are both red.

Solution

The number of ways of choosing two red cards (favourable outcomes) is ${}_{26}C_2$.
The number of ways of choosing two cards (possible outcomes) is ${}_{52}C_2$.

$$\text{probability} = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

$$= \frac{{}_{26}C_2}{{}_{52}C_2}$$

$$= \frac{\frac{26!}{(24!2!)}}{\frac{52!}{(50!2!)}}$$

$$= \frac{26!50!}{24!52!}$$

$$= \frac{25}{102}$$

Additional Examples
Mathematics 12: p. 501 (Questions 1-3)
Mathematics: Principles and Process 12: p. 416
Holtmath 12: p. 363

Exercises

4. Use permutations to solve the question in Example 2.

5. Each of the letters of the word **CANADA** is written on a slip of paper and put in a hat. What is the probability that two successive letters drawn from the hat are both A?

EXERCISE 8b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 504 (Questions 2, 4, 6)
Mathematics: Principles and Process 12: p. 417 (Questions 1, 5, 6)
Holtmath 12: p. 364 (Question 2), p. 371 (Questions 2, 3)

Multiple-Solution Approaches

Challenge yourself

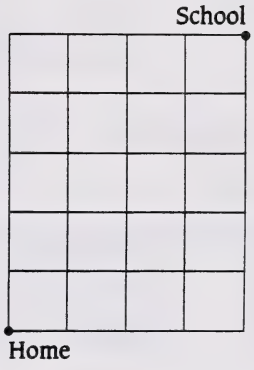
Video Segment 8c deals with a problem that can be interpreted as either a permutation situation or a combination situation. Carefully note the logic underlying the two solutions presented for the problem.

MATH FACTOR

Watch Video Segment 8c.

Example

A school is located five blocks north and four blocks east of a student's house. The blocks in the neighbourhood are laid out in a rectangular north-south and east-west pattern. If the student travels only north and east, how many routes can the student ride a bike to school?



Solution

There are at least three different methods for solving this problem. One of them involves using the diagram itself to draw all the possible routes. Two methods are used here.

Method 1: Using Combinations

Let N represent biking one block north and E represent biking one block east. In total, the student must bike nine blocks: five blocks to the north and four blocks to the east. One of these routes, NENENENEN, represents alternately biking north and then east for a total of nine blocks.

To find the total number of routes, find the number of ways of choosing the five N's from the nine blocks that must be biked.

So, the total number of routes is ${}_nC_r = {}_9C_5$

$= 126$

Method 2: Using Permutations

One possible route is NNNNNEEEE, representing biking the five north blocks first. Calculate the number of ways of arranging the five N's and the four E's.

The permutation rule for repetitions states that there are $\frac{n!}{a!b!} = \frac{9!}{5!4!} = 126$ routes.

Exercises

6. Three different five-member teams are chosen from a group of 15 athletes. In how many ways can this be done?

7. Starting at the top left corner, how many different paths are there that spell the word **PERMUTATION**?

```
P E R M U T
E R M U T A
R M U T A T
M U T A T I
U T A T I O
T A T I O N
```

EXERCISE 8c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics: Principles and Process 12: p. 425 (Question 7)
Holtmath 12: p. 360 (Question 17)

Binomial Theorem and Pascal's Triangle

Challenge yourself

Find the coefficient of the term containing x^{12} in the expansion of $(x+3y)^{17}$.

MATH FACTOR

Watch Video Segment 8d.

Binomial Theorem

In general, the expansion of a binomial can be given by the following:

$$(x+y)^n = {}_nC_0 x^n + {}_nC_1 x^{n-1} y + {}_nC_2 x^{n-2} y^2 + \ldots + {}_nC_k x^{n-k} y^k + \ldots + {}_nC_n y^k,$$

where $n \in W$

Note: The general term ${}_nC_k x^{n-k} y^k$ is the $(k+1)$th term; so, $t_{k+1} = {}_nC_k x^{n-k} y^k$.

Example

In the expansion of $\left(x^2 - \frac{1}{x}\right)^6$, find the general term and the constant term.

Solution

$$\begin{aligned}\text{general term} &= {}_nC_k x^{n-k} y^k \\ &= {}_6C_k \left(x^2\right)^{6-k} \left(x^{-1}\right)^k \\ &= {}_6C_k x^{12-2k} x^{-k} \\ &= {}_6C_k x^{12-3k}\end{aligned}$$

$\frac{1}{x}$ can be written as x^{-1}.

The constant term is the term containing x^0.

$$\begin{aligned}\therefore x^{12-3k} &= x^0 \\ 12-3k &= 0 \\ 12 &= 3k \\ k &= 4\end{aligned}$$

$$\begin{aligned}t_{4+1} &= {}_6C_4 x^0 \\ t_5 &= {}_6C_4 \\ &= 15\end{aligned}$$

The constant term is 15.